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Learning Mathematics with Mathematical Objects: Cases of Teacher-Made Mathematical Manipulatives

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Access to maker technologies has catalyzed and amplified the possibilities for creating physical materials that are responsive to the needs of students. Opportunities for design and fabrication of original mathematics manipulatives have been incorporated into the teacher education program at Montclair State University. Participating preservice elementary teachers design and make original mathematics manipulatives. Three case studies examine ways in which this process enhances students' mathematical reasoning, sense-making, and understanding. The designs created are made available through links to the open source Educational CAD Model Repository, enabling others to replicate the objects described.

EDITORS' NOTE: We are pleased to announce a new feature of the *CITE Journal* in this issue. We are establishing an open-source Educational CAD Model Repository that can be used to store designs for physical objects that can be fabricated using 3D printers, laser cutters, digital die cutters and other fabrication tools available in maker spaces. A new section of the journal, Objects to Think With: Educational Fabrication and Design, has also been added to take advantage of this new capability. This section of the journal is sponsored by the International Technology and Engineering Education Association (ITEEA), which joins the other five associations (AMTE, ASTE, ELATE / NCTE, CUFA / NCSS, and SITE) that jointly sponsor and publish the *CITE Journal*.

We are equally pleased to copublish an inaugural article that demonstrates the potential of these new capabilities: "Learning Mathematics with Mathematical Objects: Cases of Teacher-Made Mathematical Manipulatives" by Steven Greenstein and Eileen Fernández. This article describes ways in which preservice elementary teachers design and fabricate original math manipulatives in their teacher education program at Montclair State University. These case studies examine ways in which this process enhances students' mathematical reasoning, sense-making, and understanding. The authors share the designs the preservice teachers created, as well as the preservice teachers' evaluations of the objects they made based on use with students. The cases described in the paper will be of interest to CITE-Math Education readers, providing insight into creative ways to use maker technologies to support preservice teachers' awareness about use of physical objects to support student learning.

The concept of making is not a new one; "people have been making things forever" (Halverson & Sheridan, 2014, p. 495). Teachers have been making things forever, too. What is new is that broader access to digital design and fabrication technologies has both catalyzed and amplified the possibilities for making educational classroom materials that are responsive to the particular needs of its students.

Three *objects to think with* are presented in this article. These objects were designed and fabricated by future teachers of elementary mathematics with the goal of helping students learn a particular concept in a meaningful way. The teachers' evaluations of the things they made are also shared, including explanations of what their students learned as they worked with those objects. These findings are presented in the hope that they will encourage others to make these things for themselves or for another teacher or learner. Better yet, this illustration of the power of thinking with physical objects in mathematics education may inspire others to design and make their own objects to think with.

Manipulatives in Mathematics Education

Manipulatives in mathematics education are physical tools that are designed to help students learn mathematical ideas. Although concepts are not inherently visible within them, it is their designer's intention to embed those concepts in their design. For instance, a child without an understanding of place value is unlikely to see the base-10 relationship between the rods and units in a set of base-10 blocks (see Figure 1). But the intention is that as they progress through a deliberately designed sequence of activities with those blocks, the relationship becomes apparent: one "10" can be decomposed into 10 "ones"; 10 ones can be composed into one 10. Through this progression,

- physical knowledge becomes conceptual knowledge (Kamii & Housman, 2000);
- ideas are abstracted from their concrete referents (Bruner, 1966);
- material artifacts become psychological tools to think with (Verillon & Rabardel, 1995);
- engraved pieces of plastic become base-10 blocks.

These and other common classroom manipulatives, including fraction strips, pattern blocks, and connecting cubes, are shown in Figure 1.

Figure 1
Base-10 Blocks, Fraction Strips, Pattern Blocks, and Connecting Cubes

Rationales for the use of manipulatives in mathematics education tend to be grounded in a constructivist theory of learning (Piaget, 1970), since the proper use of manipulatives involves having students construct these ideas for themselves through active, hands-on manipulations of these physical tools. Indeed, as the theory goes, the power of manipulatives lies in their capacity to support the construction of abstract mathematical concepts from students' sensorimotor engagement with concrete tools (Kamii & Housman, 2000; Vygotsky, 1978).

Lately, perspectives on embodied learning (Lakoff & Núñez, 2000; Nathan, 2021; Thompson, 2007) have contributed to the discussion by extending the site of thinking beyond the head to throughout the body, theorizing the mind-body mechanisms through which cognitive structures arise from engagement with the full sensations of our experiences. Indeed, it might be hard to imagine how one's images (Tall & Vinner, 1981) of fraction and place-value concepts, for example, could *not* constitute embodied actions of dividing, iterating, grouping, and trading.

From an embodied perspective, even the most complex ideas are grounded in and emergent from the sensorimotor activity of bodily experiences in the world. Thus, manipulatives can provide the experiential context for activities essential to students' learning of mathematics. In a very real sense, *making sense* is *that which we make of our senses*. That being so, the manipulation of manipulatives constitutes the manipulation and making of one's ideas.

Froebel's Gifts: The First Manipulatives

A valuable insight may be gained from returning to the origins of manipulatives to appreciate their pedagogical potential. That is, the original designer of the physical tools that are now called manipulatives, Friedrich Froebel, had even loftier ambitions than helping students learn mathematics. Froebel was the German educator who is credited with inventing the concept of kindergarten. He believed that every child needed to be active and engaged in open and meaningful play. He also believed that every child should appreciate the harmony of forms and relationships found in nature (Provenzo, 2009), those of beauty (including symmetry, pattern, and order), knowledge (including the mathematical and scientific concepts of size, shape, and balance), and living things (including worldly things and events). Froebel assumed that there was a mathematical logic underlying these natural forms and relationships, so to cultivate children's appreciation of them, he developed a series of materials called *gifts* (see Figure 2) that he hypothesized would be useful for teaching children their underlying logics.

Figure 2
A Collection of Froebel Gifts

Froebel's gifts convey in tangible form the theory and appeal of his model of educational play, and although his work predates talk of constructivism by at least a hundred years, his theory of how students learn through gift play sounds as if it emanates from the very same source: "Perception is the beginning and the preliminary condition for thinking. One's own perceptions awaken one's own conceptions, and these awaken one's own thinking…" (Wiggin & Smith, 1895, p. 1).

In accordance with that view, Froebel sought to develop "the right forms for awakening the higher senses of the child" (p. 6) to nature's physical forms and to the connections among them. These were his *gifts*, and they include the same pattern blocks and geometric building blocks that are found in classrooms today. Artists and designers such as Le Corbusier, Frank Lloyd Wright (his mother was a Froebelian educator!), Paul Klee, Wassily Kandinsky, Buckminster Fuller, and Piet Mondrian say their aesthetic sensibilities were greatly influenced by their experiences with Froebel's gifts in kindergarten (Provenzo, 2009). In order from left to right and top to bottom, selections of their work that are reminiscent of Froebel's gifts appear in Figure 3.

Figure 3
Le Corbusier's Pavilion, Wright's Fallingwater, Klee's Red Bridge, Kandinsky's Black and Violet, Fuller's Montreal Biosphere, and Mondrian's Composition A.

What is inspiring about a recall to the work of Froebel is that it is a reminder that knowing mathematics is not the ultimate goal of learning mathematics. It certainly was not for these artists and designers. On the contrary, for them, learning the natural forms and logic of mathematics through embodied engagement with manipulatives nurtured appreciable skills and dispositions that formed the contours of their life's work. These are just some of the potential benefits students might glean from playing with manipulatives in their mathematics education.

Making Manipulatives for Teaching and Learning Mathematics

In our own work, we have explored the potential benefits that students preparing to be elementary teachers might glean from making (Halverson & Sheridan, 2014) new manipulatives. In their mathematics education coursework, the future teachers in our teacher preparation programs learn about how manipulatives can support a student's learning. Naturally, that conversation involves considerations of both mathematics and teaching. Accordingly, we thought that a pedagogically genuine, open-ended, and iterative design experience centered on the making of an original physical mathematics manipulative might inform the kinds of conceptual and pedagogical thinking that would enable these future teachers to support and promote their students' mathematical reasoning, sense-making, and understanding. We implemented this project in three sections of a mathematics content course for future teachers at Montclair State University, which is a midsized university designated as a Hispanic-serving institution in the northeastern United States.

In the remainder of this article, three examples of the manipulatives these future teachers made are presented. These presentations include the rationales for their designs, links to the relevant mathematical Common Core mathematical standards (CCSSO, 2010), and excerpts from an efficacy study implemented with a child that demonstrated the value of their manipulative for learning a mathematical idea. These manipulatives were designed in Tinkercad (Autodesk Inc., 2020) and printed with Polylactic Acid, a natural and recyclable material derived from renewable resources on Makerbot Replicator 3D printers. Links to digital printing files in an Educational CAD Model Repository are provided to enable others to replicate these objects.

Case 1: The Fraction Orange

Dolly (all student and teacher names are pseudonyms) designed a tool with affordances for the exploration of fraction concepts (CCSSM 6.NS.A.1). Her *Fraction Orange* (Figure 4) is a sphere partitioned into two hemispheres; one hemisphere is further partitioned into fourths, eighths, and 16ths of the whole; the other into sixths and 18ths. She tested her tool in a problem-solving interview with an adult named Lyle in order to assess his understanding of fraction division. The problem she gave him was to find the value of $\frac{1}{2} \div \frac{1}{4}$. The meanings that Lyle assigned to pieces of the orange are identified in the photo on the right of Figure 4.

Figure 4
Dolly's Fraction Orange

The fabrication files for the Fraction Orange can be accessed through the Educational CAD Model Repository via the following link: https://educationalmanufacturing.org/model/fraction-orange/

Lyle's initial response was to implement the flip-and-multiply algorithm for fraction division, which yielded an answer of 2. Because Dolly was interested in assessing what Lyle understood about fraction division, she asked him to find the solution with the orange next. Lyle interpreted the problem as “a half divided by a quarter” and then identified a half-piece and the two fourth-pieces that lay inside it. Lyle reasoned that the answer was 4. When Dolly pointed out that his answer did not match what his algorithm produced, Lyle experienced disequilibrium. He replied, “Uh oh. Why doesn’t that work?” At this point, Dolly realized she was not so sure of her own understanding of fraction division, either. She identified the half-piece and proposed this interpretation of the problem to Lyle: “And how many quarters go into a half?” This exchange followed:

Lyle: [Pointed to ½ on the page where he had written his work] So this is half of a whole [then pointing to ¼ on the page], and this is a quarter of a whole. [He then turned his attention to the orange and pointed to the half-piece] Half of a whole. [He pointed to each quarter-piece.] Quarter of a whole [pointing to the two quarter-pieces] is 2.

Thus, Lyle appeared to be establishing that the number of quarter-pieces he identified, 2, is the answer to the posed problem, ½ ÷ ¼.

> Dolly: [pointing to the two quarter-pieces and agreeing with Lyle] Yeah, ‘cause there’s two quarters of a whole.
> Lyle: Yeah, that makes sense.
>
> Dolly and Lyle concluded their problem-solving activity satisfied with their achievement, as indicated in their final exchange:
>
> Dolly: Woo! We did it!
> Lyle: Yeah, but it was complicated.
> Dolly: [laughing] It was.

In contrast to the procedural activity that Lyle enacted without a sense of what it meant to do so, his activity with the physical manipulative instigated a drive toward sense-making that culminated in an embodied understanding of fraction division. The orange’s affordances for the exploration of fraction division – parts of a whole that are embedded in other parts of the whole – were essential to Lyle’s sense making of the concept.

Case 2: The Minute Minis

Mia wanted to design a tool that she hoped would alleviate the anxiety that children often experience when they learn about fractions. Casey was interested in helping students learn how to tell time. When they realized that both of their designs featured partitioned circular shapes, they decided to work together to create Minute Minis (Figure 5). Minute Minis were designed to help students learn the abstract concept of time using concrete representations of fractions. Specifically, they wanted to help children in second grade use what they already know about telling time (to the nearest half hour; CCSSM 1.MD.B.3) to learn to tell time more precisely (in increments of 5 and 15 minutes; CCSSM 2.MD.C.7) while also learning the part-whole meaning of fractions (CCSSM 3.NF.A.1).

Minute Minis are essentially relabeled fraction circles: the whole in Figure 5 is labeled *1 hour*, the halves are each labeled *30 minutes*, the fourths are labeled *15 minutes*, and the 12ths are labeled *5 minutes*.

Figure 5
Casey and Mia's Minute Minis

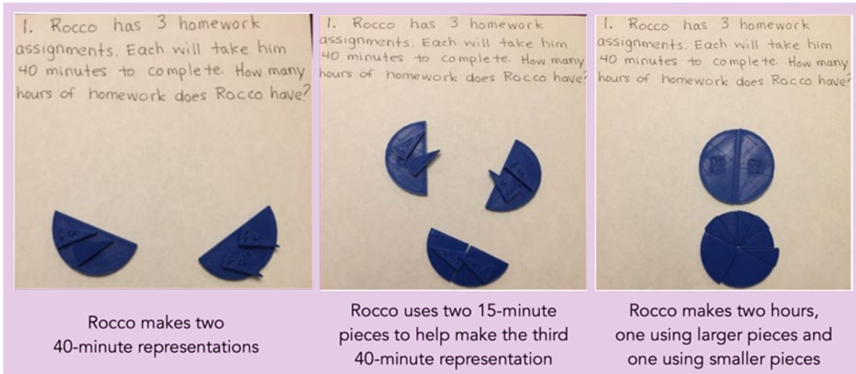
The Minute Minis fabrication files are in the Educational CAD Model Repository in several formats, including a 3D-printed version, a laser-cut version, and a cardstock version available as a PDF file. When the cardstock version is printed, students can cut out the pieces with scissors. The files in the repository are available through the following link: https://educationalmanufacturing.org/model/minute-minis/

Casey and Mia tested their manipulative with a 9-year-old named Rocco. They let him play with the Minute Minis for a while and then they gave him this task: "Rocco has 3 homework assignments. Each will take him 40 minutes to complete. How many hours of homework does Rocco have?" Rocco's initial response was, "120 minutes." Knowing that Rocco prided himself on the speed of his mental math, Casey and Mia inferred that he found his solution by multiplying 40 and 3. In response, they asked him to slow down and reread the problem. After doing so, he responded, "Oh, I can't do that." Casey and Mia conjectured that Rocco might have been confused about how to convert minutes to hours, so they invited him to use their manipulatives to figure out a solution.

First, Rocco made two groups of 40 minutes using one 30-minute piece and two 5-minute pieces per group (Figure 6, left). To make the third group of 40 minutes, he asked for another 30-minute piece. Upon learning that there were no more, he assumed he could not finish the problem. However, once prompted to see if he could make 40 minutes some other way, Rocco went back to work.

Eventually, he figured out that he could use two 15-minute pieces to make 30 minutes, and with two more 5-minute pieces, he was able to complete his third group of 40 minutes (Figure 6, center). Then, without prompting, Rocco moved the largest 30-minute pieces together to form 1 hour, and then the two 15-minute pieces and six 5-minute pieces to form a second hour (Figure 6, right). Then he announced his answer, "2 hours."

Figure 6
Rocco's Work on the Homework Task



When Casey and Mia designed their Minute Minis, they embedded them with time and fraction concepts. They did so because they hypothesized that their design would elicit a child's thinking about those concepts so that they could further develop it. What Rocco's problem solving revealed is that their design worked. That is, his manipulations of the Minute Minis opened a window into his understandings of time in terms of the parts (or minutes) of a whole (hour). And as evidenced by the apparent cognitive demand of the task and Rocco's overcoming an obstacle on his path to a solution, those manipulations also supported the development of ideas about time and fractions that were unknown even to him.

Case 3: The Decimal Snake

Roda designed a tool that she wanted to use to teach a child about decimals and decimal comparison (CCSSM 5.NBT.A.3). Her Decimal Snake (Figure 7) consists of 10 connected pieces, each of which is equally partitioned into 10 parts. Thus, the snake can be used to represent 10ths of 10ths (or 100ths) of a whole (i.e., any decimal value between 0.01 and 1). These design features constitute concepts of the whole and its decimal parts that Roda embedded into her design.

Figure 7
Roda's Decimal Snake

The fabrication files for the Decimal Snake can be accessed through the Educational CAD Model Repository via the following link:
https://educationalmanufacturing.org/model/decimal-snake/

Roda engaged a child named Greg in a problem-solving interview to evaluate the efficacy of her manipulative. At one point in the interview, she asked Greg to compare 5.5 and 5.47. (Note that it would not be possible to represent 5.47 given that the entire snake represents 1.) Greg responded, "5.47 is 5 and 47 hundredths, because it's 3 hundredths away from 5 and 5 tenths." Because Roda was interested in assessing how well her tool can support Greg's reasoning, she then asked him, "Use the tool to show me?"

For the next 60 seconds, Greg struggled to locate 5.5 and 5.47 on the snake. Finally, he located 5.5 at what we would identify as 0.55; and then he located 5.47 at 0.47 (see Figure 8). Given that several minutes earlier Greg established that the entire snake is the "whole" and that each piece of the snake is one 10th of a whole, we inferred from his solution – locating 5.5 at 0.55, and 5.47 at 0.47 – that he had unintentionally designated each piece of the snake as 1 (as opposed to 0.1) and each partition of a piece as 0.1 (as opposed to 0.01). In doing so, he changed his designation of the entire snake from the whole (1) to 10, and consequently, each piece of the snake then represented 1. Thus, 5.5 would be located in the fifth partition of the fifth piece.

Figure 8
Greg Locating 5.5 and 5.47

Roda's next move aimed to help Greg identify and resolve his confusion. When she asked him to "show me one 10th," he pointed to the first 10th piece. When she asked for "two 10ths," he pointed to the second 10th piece. Then she asked, "Where is 5 and 5 10ths?" In doing so, she perturbed his thinking and provoked disequilibrium. Soon thereafter, Greg resolved it and declared, "Oh, wait! This [entire snake] is one whole! 5 and 5 10ths, you can't even make it out of the snake!"

In response to this unanticipated move in Greg's activity, Roda leveraged an affordance of her tool – namely that each piece of the snake could represent either a 10th of a whole or one of 10 wholes – and exploited it to support new ways of thinking for Greg as he resolved his confusion about the representational capacities of the tool.

> Roda: You need how many snakes to make 5.5?
> Greg: You need 5 – No, 6 snakes!
> Roda: How can we compare 5.5 and 5.47 using 1 snake? Is that possible?
> Greg: We can pretend that each piece is one snake.

In this instance, Roda leveraged a conceptual design decision she embedded in her manipulative that enabled the user of the decimal snake to engage in conversations about the unit whole and its decimal parts. That decision allowed for flexibility in naming the unit whole in relation to the whole snake and its constituent pieces. Roda's rationale for leveraging that feature of her design was a pedagogical one. Rather than correct Greg's interpretation, she helped him reason through his interpretations to resolve the confusion for himself.

In this respect, the snake's capacity for flexible interpretations of quantities (i.e., pieces of the snake could be regarded as wholes, 10ths, and so forth) supported Roda's intention to use the tool to reveal and respond to Greg's thinking, a practice she values as a mathematics teacher. Worth noting, Roda did not plan for this conversation about the unit whole, nor had she anticipated it. Regardless, the manipulative she designed enabled her to do so.

Conclusion

The intent of this article was to convey "images of the possible" (Shulman, 2004; p. 147) regarding the learning potential of things to think with in mathematics education. To spark the reader's imagination about what might be possible, the foundational work of Froebel was described. Froebel designed manipulatives to nurture children's appreciations of beauty, logic, and living things. Then, this space of natural forms and relationships was used to consider the case of teaching and learning mathematics. Three vignettes were presented in which teachers effectively used manipulatives they designed to help them teach a mathematical concept to a learner. In each of these vignettes, a learner was actively engaged in meaningful, mathematical play, just as Froebel and many others have endorsed (e.g., De Holton et al., 2001; Piaget, 1962; Steffe & Wiegel, 1994; Vygotsky, 1978).

Amidst each learner's play, they encountered a moment of disequilibrium. With the support of a teacher, the learner leveraged mathematical relationships embedded by the teacher within their tool to resolve the disequilibrium and persevere to learn mathematics. The flexible designs of the Fraction Orange and Decimal Snake enabled learners to freely assign their personal mathematical meanings to their attributes (i.e., the "half" of the Orange and the "whole" of the Snake). Subsequently, the learners relied on mathematical relationships among the parts of the manipulative that were consequential to those assignments to solve the problem posed to them and, thereby, learn something new about fractions or decimals.

In the case of the Minute Minis, the design constraint of too few 30-minute pieces required the learner to find an equivalent alternative (i.e., the substitution of two 15-minute pieces), which he did. This discovery, in turn, prompted valuable rearrangements of parts and wholes that culminated in a solution to the problem and in the child making a connection between the part-whole meaning of fractions and the part-whole units of time.

These findings are presented as an inspiration for others to try out these manipulatives in their own classrooms. Tinkercad tutorials and other resources to support this kind of work are available on the Re-making Mathematics website: teachermakers.montclair.edu. These new design and fabrication capabilities make it possible for teachers to design and make their own manipulatives to meet the special and particular needs of their students.

References

Autodesk Inc. (2020). *Tinkercad* [Computer software]. https://www.tinkercad.com/

Bruner, J. S. (1966). *Toward a theory of instruction*. Belknap Press.

De Holton, D., Ahmed, A., Williams, H., & Hill, C. (2001). On the importance of mathematical play. *International Journal of Mathematical Education in Science and Technology, 32*(3), 401-415. https://doi.org/10.1080/00207390118654

Halverson, E. R., & Sheridan, K. M. (2014). The maker movement in education. *Harvard Educational Review, 84*(4), 495-504, 563, 565.

Kamii, C., & Housman, L. B. (2000). *Young children reinvent arithmetic: Implications of Piaget's theory*. Teachers College Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Nathan, M. J. (2021). *Foundations of embodied learning: A paradigm for education*. Routledge. https:/doi.org/10.4324/9780429329098

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Authors. http://corestandards.org/

Piaget, J. (1962). *Play, dreams, and imitation in childhood* (C. Gattegno & F. M. Hodgson, Trans.). W. W. Norton & Co., Inc.

Piaget, J. (1970). *Genetic epistemology*. Columbia University Press.

Provenzo, E. F., Jr. (2009). Friedrich Froebel's gifts: Connecting the spiritual and aesthetic to the real world of play and learning. *American Journal of Play, 2*(1), 85-99.

Shulman, L. (2004). *The wisdom of practice: Essays on teaching, learning, and learning to teach*. Jossey-Bass.

Steffe, L. P., & Wiegel, H. G. (1994). Cognitive play and mathematical learning in computer microworlds. *Educational Studies in Mathematics, 26*(2), 111-134. http://dx.doi.org/10.1007/BF01273660

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169. https://doi.org/10.1007/BF00305619

Thompson, E. (2007). *Mind in life: Biology, phenomenology, and the sciences of mind*. Harvard University Press.

Verillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education, 10*(1), 77-101. www.jstor.org/stable/23420087

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Harvard University Press.

Wiggin, K. D., & Smith, N. A. (1895). *The republic of childhood: Froebel's gifts*. Houghton, Mifflin, and Company.