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Theoretically Framing the Pedagogy of Learning to Teach Mathematics With Technology

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Frameworks can influence the work of mathematics teacher educators (MTEs) in many different ways. Frameworks can suggest a structure around which MTEs design instruction, provide a common language for communicating with prospective teachers, and support prospective teachers as they design their own instruction. This paper reports findings related to the frameworks that MTEs are currently using in their work of preparing secondary mathematics teachers to teach with technology. Findings include a list of 17 frameworks, which fall into four categories with respect to their framing: (a) how students use and learn with technology, (b) the design and evaluation of technology tools and tasks, (c) how teachers use technology, and (d) how teachers learn to use technology. The individual frameworks within each category are discussed and implications for mathematics teacher educators are presented. Implications include a critical discussion of what is missing among the frameworks and challenges for the field.

Frameworks can influence the work of mathematics teacher educators (MTEs) in many different ways. Frameworks can suggest a structure around which MTEs design instruction, provide a common language for communicating with prospective teachers, and support prospective teachers as they design their own instruction. The Association of Mathematics Teacher Educators (AMTE, 2017) *Standards for Preparing Teachers of Mathematics* (SPTM) articulated the importance of frameworks in mathematics teacher education in indicator P.3.4, stating that

an effective mathematics teacher preparation program ensures that practice-based experiences, including mathematics methods courses and equivalent learning experiences, provide candidates with experiences using tools and frameworks grounded in research to develop core pedagogical practices and pedagogical content knowledge for teaching mathematics. (p. 35)

The document went on to explain that frameworks serve as “important vehicles for connecting theory and practice and guiding candidates in their work with authentic artifacts of teaching” (p. 36). Frameworks are thus positioned as important to the work of MTEs.

The field of mathematics teacher education has begun to consider seriously the ways and the extent to which MTEs follow such recommendations – how MTEs use research-based frameworks to build and explore our work with prospective mathematics teachers (PSMTs; e.g., see Kastberg et al., 2012, Kastberg et al, 2017). In 2005 Mewborn emphasized this work when she challenged the field to develop conceptual definitions for big ideas and frameworks for “long-used ideas” (p. 7) in mathematics teacher education.

With the articulation of new frameworks, however, comes the need to compare and contrast them to identify gaps and overlap, and the need to reconcile differences between those that rival each other (Lester, 2005). In doing so, the field need not coalesce around a single framework, but rather can adopt and adapt those that meet our goals (Lester, 2005).

If MTEs aim to meet the goals set forth by the AMTE SPTM, specifically as they relate to teaching mathematics with technology (C.1.6), the time is ripe for this careful consideration of the frameworks that have emerged related to teaching with technology. This paper describes our findings related to the frameworks that MTEs are currently using in their work of preparing secondary mathematics teachers to teach with technology.

Framing Our View of Frameworks

We adopted a broad view of frameworks for this work and speak of these frameworks in terms of what is being framed. Many phenomena are related to the pedagogy of learning to teach mathematics with technology that could be framed (i.e., viewed in a particular way). One might seek to

frame the work of the MTE, the work of the mathematics teacher, or the work of the mathematics students. One might seek to frame technology itself.

Regardless of its object, framing takes an explicit theoretical stance regarding ways of viewing this object – how one might distinguish this phenomenon from other phenomena, how one might discern critical characteristics, and nuances of the phenomenon. Noting the importance of such work, Leatham (2019) stated, “We frame our work theoretically by drawing on, adapting, and developing theories (plausible explanations for phenomena) in order to delineate the scope and nature of the phenomena we study” (p. 169).

With this conceptualization of framing in mind, in this study we sought to answer the following research questions:

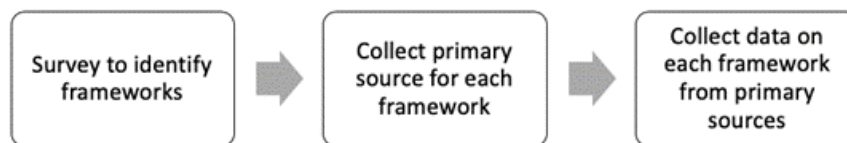
- What frameworks are MTEs using to frame their work related to teaching secondary mathematics teachers to teach with technology?
- What phenomena related to teaching mathematics with technology are these frameworks framing?

After the results of this study is a discussion of the findings. This discussion includes an examination of the sufficiency of these frameworks for helping frame these phenomena and questioning whether important phenomena might be related to learning to teach mathematics with technology that yet lack sufficient framing.

Methods

A set of frameworks identified by MTEs were the impetus for this study. These frameworks were elicited from a nationwide survey study focused on describing how secondary mathematics teacher preparation programs are preparing teachers to teach mathematics with technology. Primary sources for the frameworks elicited from the survey became the data sources for the current study (see Figure 1). Using qualitative methods these data were analyzed with respect to the research questions. Our specific data collection and analysis processes are described in detail in the sections that follow.

Figure 1 Methods for Data Collection



The Survey

Our survey was designed to elicit descriptive information about courses designed to address learning to teach mathematics with technology. Items

targeted the structure of the course or courses, course learning objectives, different types of technology utilized and frequency of use, and types of learning activities (for a full report of this study see McCulloch et al., 2021). One survey item asked participants to “list any frameworks related to teaching mathematics with technology that are explicitly shared with PSMTs in their course(s).” Responses to this item are the focus of this paper.

Since our goal was to describe how teachers are being preparing to teach mathematics with technology widely, we aimed to reach all 4-year-university-affiliated secondary mathematics teacher preparation programs. To do so, we identified all accredited university secondary mathematics teacher preparation programs by visiting the department of public instruction websites for all 50 states. (We define secondary mathematics teacher preparation programs as programs that prepare PSTs to teach Grades 9-12. In some states designation includes some or all of Grades 6-8.)

We then verified that each university on the list had an undergraduate secondary mathematics teacher preparation program and identified an appropriate contact person. This search resulted in a list of 956 accredited programs. Using a presurvey email we confirmed the correct contact for 383 of the 956 (40%) accredited programs prior to survey distribution. The survey was sent to all 956 universities along with at least one reminder email if necessary, resulting in two to five email contacts per program.

The response rate was 30% overall, and 61% from those programs for which an appropriate contact had been confirmed. The overall response rate is larger than that of previous studies focused on this topic (Kersaint et al., 2003; Leatham, 2006), and the respondents represent a broad cross-section of universities from 48 of the 50 states and from different Carnegie classifications.

Of the 286 programs that responded to the survey, 61 (21%) responded to the item about shared frameworks (i.e., “Which frameworks related to technology, if any, do you share explicitly with students in these classes?”). These responses yielded a list of 23 different frameworks, ranging from theoretical frameworks (e.g., Technology Pedagogy and Content Knowledge [TPACK]), to state-specific standards (e.g., Ohio Learning Standards for Technology), to position statements from professional organizations (e.g., the National Council of Teachers of Mathematics [NCTM] 2015 Technology Position Statement).

Data Sources for the Frameworks

With a set of frameworks identified, we then set out to collect data related to each. In this situation, the data were collected from primary sources for each identified framework. For example, we noted how the framework was developed, for whom (e.g., teacher educators or practicing teachers), any figures used to represent it, and whether it was peer reviewed. Data collection was facilitated by the use of a Google form. The form provided prompts to guide the extraction of data from the sources and recorded and organized responses to these prompts in a spreadsheet.

Finally, precise summaries of each framework were constructed. Summary construction was an iterative process. First, a pair of research team members worked collaboratively to write a one- to two-page summary from the primary source. Based on this first iteration the team agreed upon the important features to include a clear description of the framework (including any figures), its original purpose (e.g., to inform or to describe), and identification of any work from which it is built.

Next, two researchers reviewed the summaries to make sure that all of the previously mentioned features were included. Any summaries that did not include all of those features were revised. Then the entire research team read each summary for consistency with the primary resources with any further refinements done as a group. At this time the summaries were approximately one page each. In an effort to focus on the goal of each of the frameworks (i.e., the phenomenon it is framing), summaries were abbreviated to brief descriptions that left out details about where the framework originated and how it was developed and only focused on what it framed. For example, our brief description of Pea's (1985) framework stated,

Computers and technology are often viewed as *amplifiers* of cognition, in that they allow humans to perform tasks with increased speed and accuracy. Pea claimed that computers and technology can also act as *reorganizers* of cognition, in that technology can transform the way in which an individual views a concept or idea.

While teams of two led the construction of each summary and brief description, each was brought back to the team to work to ensure what was written aligned with the original resources. This process of refinement continued until we were confident both the summary and the brief description accurately reflected the original author's work. In addition, one draft of the brief descriptions of each framework was vetted by other mathematics teacher educators in a conference session of the AMTE (Bailey et al., 2020). Feedback from that session, mostly focused on clarity not content, led to further refinement of the brief descriptions. (The framework summary cards can be downloaded from <https://ptmt.s3.amazonaws.com/impact/Framework%20Cards.pdf>)

Analysis

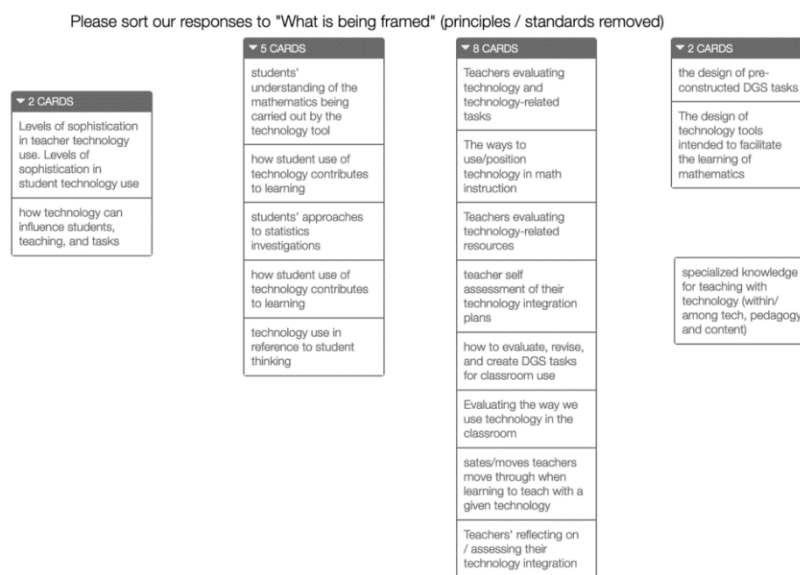
As expected, the survey prompt elicited a variety of responses, including some frameworks that were not specific to technology and some that were not aligned with the conceptualization of framing guiding this study. As an example of the latter, the Ohio Learning Standards indicate the expectation to integrate technology in instruction, but they do not frame a phenomenon related to learning or teaching mathematics with technology. To address this issue we removed the survey responses that did not meet the criterion of explicitly framing a phenomenon related to learning or teaching mathematics with technology ($N = 6$). This approach resulted in a set of 17 frameworks.

We open-coded the 17 framework brief descriptions to identify themes related to what was being framed by each. This coding was done through

a sorting process in which framework descriptions (with no identifying features) were placed on virtual cards (using the card sort capabilities of the software program Desmos Activity Builder), and each member of the team sorted the cards into categories utilizing the guiding question, “What is being framed?”

The Desmos teacher dashboard “overview” feature for card sorts was utilized to identify commonalities among the sorting done by each member of the research team and to identify overarching themes among the frameworks (see Figures 2 and 3). (Our sorting allowed for frameworks to be placed in more than one category if appropriate.) This refining process resulted in four themes, each of which will be described in detail in the findings.

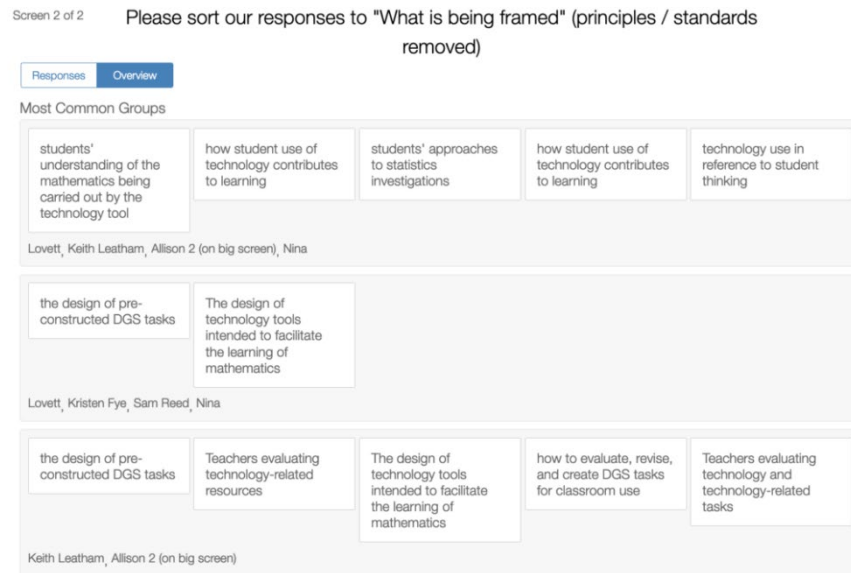
Figure 2 Sample Desmos Card Sort



Findings

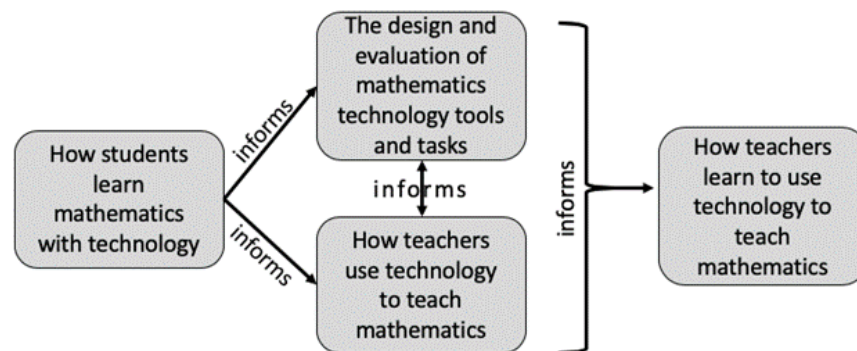
The 17 frameworks included in our analysis are, in essence, the answer to our first research question (What frameworks are MTEs using to frame their work related to teaching secondary mathematics teachers to teach with technology?). Of these 17 frameworks, all but one, the Substitution, Augmentation, Modification, and Redefinition (SAMR) model, have been peer reviewed. Examining the audience each was intended for, we found that 12 of the frameworks were derived from mathematics-specific contexts, with the remaining 5 originating from a broader perspective. The two most commonly submitted frameworks, TPACK ($N = 13$) and the SAMR model ($N = 8$) are not from mathematics-specific contexts.

Figure 3 Overview of the Sample Desmos Card Sort



By way of answering our second research question, the four resulting “what is being framed” categories are described (see Figure 4), beginning with the consideration of how students use and learn with technology. This category informs both the design and evaluation of technology tools and tasks and ways teachers use technology in instruction. These categories, in turn, inform the framing of how teachers learn to use technology for instruction.

Figure 4 Four Framing Categories Related to Preparing Teachers to Teach Mathematics With Technology



Framing How Students Learn Mathematics With Technology

This category, framing how students learn mathematics with technology, brings together a collection of frameworks that can be used to describe

how students use technology in mathematics courses, the ways in which students learn with technology, and other cognitive dimensions of students' technology use. Within this category, three of the five frameworks originated in mathematics-specific contexts (virtual cultural theory; master, servant, partner, extension of self, and white box/black box), and two frameworks are not mathematics-specific (activity theory and amplifier/reorganizer)

Virtual Cultural Theory

Shaffer and Kaput's (1998) virtual cultural theory was based on the power of "computational media" (a term meant to better depict the impact of future computers to be transformational) to externalize, or cognitively off-load, significant pieces of algorithmic processing. By doing so, such technology affords the power to transform the way we approach mathematics. In a virtual culture, students will have innovative ways of sharing new forms of mathematical experiences, mathematical representations, and mathematical understanding. "The implications of virtual culture for mathematics education will be an increased emphasis on embedded and situated mathematics – on mathematics as a way of knowing the world rather than on mathematics as computation or mathematics as formal proof" (Shaffer & Kaput, 1998, p. 113).

Virtual cultural theory suggested that, as the nature of technology evolves, the nature of mathematical learning (and of mathematics itself) will also evolve. The most direct applications of the theory are to student use of technology to access and explore mathematics. As with other such theories, these possibilities have pedagogical implications for teachers who seek to allow or encourage their students to use technology in these ways.

Master, Servant, Partner, Extension-of-Self

Goos et al. (2000) analyzed teachers' use of technology within mathematics lessons to theorize "four roles for the interaction between teacher and technology" (p. 307): Master, Servant, Partner, and Extension-of-self. We categorized this framework here (as well as in "how teachers use technology" category) because, as this work progressed, Goos et al. (2003) came to view these roles as metaphors that "are suggestive of different ways in which *teachers and students* [emphasis added] might appropriate technology into classroom mathematical practices" (p. 76).

These roles are viewed as hierarchical with respect to user knowledge and the potential use of a given technology: Technology as (a) Master - characterized by limited knowledge of a technology and dependent technology use; (b) Servant - some knowledge of a technology, but use is limited to supporting typical ways of thinking/working without technology; (c) Partner - increased knowledge of a technology and its use enhances mathematical learning; and (d) Extension-of-self - using mathematical and communication technologies as a natural part of mathematical work.

White Box/Black Box

Buchberger (1990) used a white box/black box metaphor to describe two contrasting ways of using technology to carry out mathematical calculations. Technology is a black box if the user does not already have an understanding of the mathematical calculations being carried out by the technology. By contrast, technology is a white box if the user does have that preexisting understanding. He proposed that, as a general principle, both teachers and students should use technology as a white box.

The field of mathematics education has viewed the metaphor itself as much more valuable than the proposed didactic principle (particularly, given the evidence that students can indeed develop robust understanding of a concept without being able to carry out associated procedures by hand; see, e.g., de Beer et al., 2017; Heid, 1988). The metaphor facilitates consideration of the affordances and constraints of using technology before, while, and after coming to understand mathematical ideas. For example, such consideration has led to expanding the dichotomy to thinking of what using technology as a “grey box” might mean (Cedillo & Kieran, 2003).

Activity Theory

Activity Theory is a collection of principles related to human activity that grew out of the work of Vygotsky (1978) and Leont’ev (1978; see also Engström et al., 1999; Nardi, 1996). It is sometimes referred to as tool oriented because of the central role that tools play in the theory as a mediating means. As explained by Liang et al. (2009), Activity Theory

uses activity as the basic unit for studying human practices and highlights the idea that the relationship between the subject and the object is not direct but rather mediated through the use of a tool. A tool can be something physical (e.g., wirelessly interconnected handhelds) or intellectual (e.g., rules and roles displayed on handhelds). (p. 393)

As a tool-oriented theory of learning, Activity Theory can be used as a framework for explaining how students learn mathematics using technology (see Nardi, 1996). As with other theories of learning, Activity Theory is not a theory of teaching, but it does have implications for designing learning environments.

Amplifier and Reorganizer

Pea (1985) described two metaphors to characterize technology use, *amplifier* and *reorganizer*. When used as an amplifier, technology is used to make students’ work more efficient and accurate by quickly performing calculations or computations and generating representations, but the nature of students’ thinking is not modified. In contrast, technology use as a reorganizer leverages the affordances of a given technology to shift students’ thinking in a way that would be difficult or impossible to achieve without technology. Pea’s metaphors focus on the cognitive processes of the learner (student), thus framing both the ways students learn with technology and how the affordances of technology relate to these ways.

Each of these frameworks can be used to explain some aspect of the role technology plays in technology-mediated mathematics learning. That said, not all of these theories are theories of learning per se. Some of these theories frame the nature of the activity (i.e., technology use) itself more so than the learning that may or may not result from that activity. Nevertheless, each of these frameworks has important implications for how teachers make decisions about the inclusion of technology within mathematics instruction. Knowing how students may use and learn with mathematics can inform the design of students' opportunities to learn through the use of technology.

Framing the Design and Evaluation of Mathematics Technology Tools and Tasks

In the second category of frameworks, we identified those that frame *the design and evaluation of mathematics technology tools and tasks*. These frameworks are intended to guide teachers in the creation or curation of technology learning experiences or the evaluation of technology tools for use in secondary mathematics instruction.

The five frameworks in this category ranged from framing the design or evaluation of specific types of technology tools (e.g., dynamic geometry tools) to digital instructional resources, in general. All the frameworks in this category were developed in a mathematical context. In this category, the Design Principles for Pre-Constructed Dynamic Geometry Sketches framework is presented first to provide a description of how it influenced two other frameworks: Interactive Geometry Software Framework (IGS FW) and Dynamic Geometry Task Analysis.

Design Principles for Pre-Constructed Dynamic Geometry Sketches

Sinclair (2003) utilized preconstructed geometry sketches within tasks and described elements of the task, directions, and questions that afforded or constrained the students. Specifically, she studied student engagement with such tasks from two perspectives: (a) how students responded to particular question types and sketch provisions and (b) how particular results highlighted elements that were missing (and needed) or poorly designed. As a result of this work Sinclair offered up five design principles for activities that utilize dynamic geometry sketches (see Figure 5).

Interactive Geometry Software Framework

Sherman and Cayton (2015) applied a research-based framework to a specific task and discussed how it can be used to evaluate and revise the task with respect to goals of student thinking. This particular framework was inspired by two studies: Sherman (2014), which examined how the use of technology was related to students' mathematical thinking by examining secondary school mathematics teachers' use of technology, and Cayton (2012), which used one-to-one Algebra 1 classes to study how teacher implementation of preconstructed interactive tasks influenced cognitive demand.

Figure 5 Design Principles for Preconstructed Dynamic Geometry Sketches

Design principles for activities that utilize dynamic geometry sketches:	
1.	When a question aims to focus student attention, the sketch must provide the visual stimulus. (i.e., draw attention through color, motion, and markings).
2.	When a statement prompts action (e.g., asking students to drag, observe or deduce), the sketch must provide affordances so that the student can take the required steps.
3.	When a question invites exploration, the sketch must provide alternate paths.
4.	When exploration might lead to a surprise, the sketch must support experimentation to unmask the confusion. It must be flexible enough to help students examine cases, yet constrained enough to prevent frustration.
5.	When questions are intended to check understanding, allowing students to collaborate with a sketch (i.e., a shared image) to consider and discuss aides the process.

Note. Adapted from “Some Implications of the Results of a Case Study for the Design of Pre-constructed, Dynamic Geometry Sketches and Accompanying Materials,” by M.P. Sinclair, 2003, *Educational Studies in Mathematics*, 52(3), p. 312 (<https://doi.org/10.1023/A:1024305603330>). Copyright 2003 by Kluwer Academic Publishers.

The Interactive Geometry Software (IGS) Framework combines Pea’s (1985) characterization of amplifier vs. reorganizer and Sinclair’s (2003) design principles in a two-dimensional array (as seen in Figure 6) to facilitate analysis of the use of interactive geometry within a task with respect to the goals of meaningful observation, exploration, and conjecture.

Dynamic Geometry Task Analysis

The Dynamic Geometry Task Analysis framework (Trocki, 2014), seen in Figure 7, outlines two dimensions for analyzing dynamic geometry tasks: (a) mathematical depth, drawing on cognitive demand (Smith & Stein, 1998), and (b) types of technological action, drawing on Sinclair (2003). These dimensions capture the extent to which a prompt or sketch allows for mathematical depth and the types of technological action required.

Trocki and Hollebrands (2018) classified a high-quality dynamic geometry mathematics task as one in which both the mathematical prompts and the technological actions combine to concentrate students on understanding concepts, processes, and relationships. A medium-level task contains a collection of prompts that coordinate mathematical depth and technological actions in a way that may encourage but does not necessitate that students reach generalized conclusions based on emergent invariant relationships that go beyond a static sketch. A low-level task does not contain a collection of prompts that coordinate mathematical depth and technological actions in a way that requires students to make generalized conclusions based on emergent invariant relationships that go beyond a static sketch.

Figure 6 IGS Framework

		Technology Used As	
		Amplifier Students could achieve the same goal without the technology	Reorganizer The mathematical goal of the task would be difficult to achieve without DGS
Goals	Questions	Use of Design Principles	
Make mathematically meaningful observations; look for invariant relationships.	Do the sketch and prompts use the dynamic affordances of DGS in a way that would be difficult or impossible to replicate without it?	Students create multiple static examples, either by construction or dragging, and reason from those static examples. For example students are prompted to make observations or generalizations based on a table or static measurements without reference to the sketch.	The sketch allows for continuous dragging, and students are guided to examine measurements or relationships dynamically. Students are required to make or explain observations or generalizations dynamically in terms of the sketch.
Mathematical exploration; use appropriate tools strategically.	How does technology support mathematical exploration?	Sketch and prompts guide students to investigate the same example or set of examples to explore mathematical connections or invariances. Freedom with respect to dragging does not provide alternative paths if students are all investigating the same example.	Sketch and prompts allow students to explore based on individual observation of mathematical concepts, connections or invariances within the sketch. The sketch supports students' mathematical exploration by providing alternate paths.
Make and test conjectures; modify thinking; foster curiosity.	Does the sketch provide feedback? Do the prompts encourage or require students to use feedback?	Sketch is limited by restrictive construction or does not provide feedback to allow students to explore their conjectures. Prompts do not explicitly guide students to test conjectures.	Sketch provides feedback or allows students to test and refine conjectures. Prompts explicitly guide students to use the sketch to test conjectures.

Note. Adapted from "Using Appropriate Tools Strategically for Instruction," by M. Sherman and C. Cayton, 2015, *Mathematics Teacher*, 109(4), p. 308 (<https://doi.org/10.5951/mathteacher.109.4.0306>). Copyright 2015 by National Council of Teachers of Mathematics.

Figure 7 Dynamic Geometry Task Analysis

Allowance for Mathematical Depth		Types of Technological Action	
Hierarchical Levels	Descriptions	Affordances	Descriptions
N/A	Prompt requires a technology task with no focus on mathematics.	N/A	Prompt requires no drawing, construction, measurement, or manipulation of current sketch.
0	Sketch does not have mathematical fidelity required to respond to prompt.	A	Prompt requires drawing within current sketch.
1	Prompt requires student to recall a mathematics fact, rule, formula, or definition.	B	Prompt requires measurement within current sketch.
2	Prompt requires student to report information from the construction. The student is not expected to provide an explanation.	C	Prompt requires construction within current sketch.
3	Prompt requires student to consider the mathematical concepts, processes, or relationships in the current sketch.	D	Prompt requires dragging or use of other dynamic aspects of the sketch.
4	Prompt requires student to explain the mathematical concepts, processes, or relationships in the current sketch.	E	Prompt requires the creation or consideration of multiple examples from which one can generalize.
5	Prompt requires student to go beyond the current construction and generalize mathematical concepts, processes, or relationships.	F	Prompt requires a manipulation of the sketch that allows for recognition of emergent invariant relationship(s) or pattern(s) among or within geometrical object(s).
		G	Prompt requires manipulation of the sketch that may surprise one exploring the relationships represented or cause one to refine thinking based on themes within the surprise.

Note. Adapted from “Evaluating and Writing Dynamic Geometry Tasks,” by A. Trocki, 2014, 107(9), p. 703 (<https://doi.org/10.5951/mathteacher.107.9.0701>). Copyright 2014 by National Council of Teachers of Mathematics.

Pedagogical, Mathematical, and Cognitive Fidelity

Dick (2008) proposed three types of fidelity (pedagogical, mathematical, and cognitive) that could be used to evaluate how true the design of technological tools is to the learning and teaching of mathematics those tools were designed to facilitate. Pedagogical fidelity is characterized by the extent to which students are likely to view a technological tool as “(a) facilitating the creation of mathematical objects, (b) allowing mathematical actions on those objects, and (c) providing clear evidence of the consequences of those actions” (p. 334). In other words, technological tools are pedagogically faithful when their interface clearly and transparently allows the user to enact and perceive mathematical actions, a primary pedagogical intent of teaching mathematics with technological tools.

Mathematical fidelity is characterized by the accuracy and comprehensiveness of a tool with respect to the properties and definitions of whatever mathematical concepts the tool is designed to represent. Technological tools are mathematically faithful when the technological representations of and actions on a concept are in harmony with the way that concept is viewed by the mathematics community. Students need to be able to use the tool in ways that are consistent with the ways they construct the mathematics.

Cognitive Fidelity is characterized by the extent to which the actions the tool allows align with the cognitive actions of the user. Tools are cognitively faithful when they enable students to enact explicitly with the tool various actions or schemas that are associated with a given mathematical concept.

Framework for Evaluating Digital Instructional Materials

Building on both the Replacement, Amplification, and Transformation (RAT) framework (Hughes et al., 2006) and NCTM's (2014) eight effective mathematics teaching practices, Thomas and Edson (2017, 2019) introduced the research-based Digital Instructional Materials (DIM) framework to help teachers evaluate digital resources for use in the classroom. They conceptualized digital instructional materials to be at the cross-section of curriculum materials and technology and to include the full range of digital forms of instructional resources, from digital textbooks to mathematics action technologies. Using a two-dimensional matrix (for an excerpt, see Figure 8), the DIM framework describes what each category of technology use would look like for each of the eight teaching practices.

The intent of the DIM framework is to guide teachers in their technology integration choices. This newly developed framework was found to be useful by the K-8 teachers that were part of its development, but to date it has not been tested more widely.

The five frameworks presented in this category may be effective in helping teachers choose and design technology-based tasks. In that same regard, it could show value for MTEs by providing a lens for preservice teachers to evaluate what constitutes high-quality technology tools and tasks. Excluding the Design Principles for Pre-Constructed Dynamic Geometry Sketches Framework, all in this category contain criteria about the technology's ability to maintain mathematical fidelity and keep mathematical goals at the forefront of the task. This thread of common criteria suggests that MTEs could utilize any of these frameworks to ensure teachers keep mathematical fidelity as a focus when curating and designing technology-based tasks.

Framing How Teachers use Technology to Teach Mathematics

The third category of frameworks we identified were those that frame *how teachers use technology* during mathematics instruction and evaluate that use. These six frameworks were designed to aid teachers in planning technology integration and in assessing how they are using technology within their classrooms. The first three frameworks (the Didactic Tetrahedron; Master, Servant, Partner, Extension of Self; Orchestration Types) were derived from mathematics-specific contexts; the remaining three frameworks (PIC-RAT, SAMR, and TIP) were derived from the broader context of learning and teaching with technology.

Figure 8 Excerpt of the DIM Framework

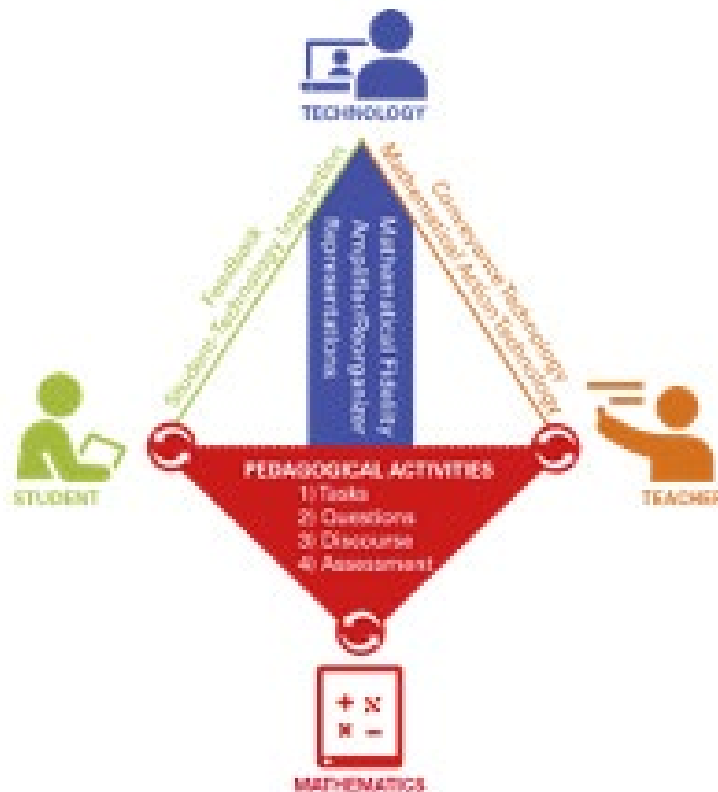
		Integration of technology use from RAT framework (Hughes, Thomas, & Scharber, 2006)		
		Replace	Amplify	Transform
Eight Effective Teaching Practices (NCTM, 2014)	Establish mathematics goals to focus learning	<ul style="list-style-type: none"> Display data, learning targets, and class information digitally Show students the "plan" for the week, including objectives and the big picture Project goals and objectives with technology instead of posting them on the board or having students write them <p>Possible Technologies:</p> <ul style="list-style-type: none"> planbook.com 	<ul style="list-style-type: none"> Use videos to launch lessons Show students and teachers the learning goals Keep track of the progress of students on each slide to get closer to the learning target State or explain the objective and goal for the lesson <p>Possible Technologies:</p> <ul style="list-style-type: none"> CCSSM Look-for app Xtramath PowerPoint Keynote Educreations Show me 	<ul style="list-style-type: none"> The tool or device adds to or changes the goals of the learning Goals are updated or changed based on individual student progress Students assess themselves before, during, and after the lesson to guide instruction Have students look at lesson or objective and then write what they think they are learning that day Have students create their own goals <p>Possible Technologies:</p> <ul style="list-style-type: none"> Google form
	Implement tasks that promote reasoning and problem solving	<ul style="list-style-type: none"> PDF or static screen rendering of textbook pages or worksheets <p>Possible Technologies:</p> <ul style="list-style-type: none"> Whiteboard app Online computational games or skills 	<ul style="list-style-type: none"> Web tools to investigate and present solutions to tasks Teacher shows instructional video that explains concept being taught <p>Possible Technologies:</p> <ul style="list-style-type: none"> Tiggly Osmos MathTwitterBlogosphere Interactive whiteboard apps LearnZillion 	<ul style="list-style-type: none"> Student investigates videos to launch lessons or presents problems Student leads video recording of work on device Use what was created with whiteboard app to provoke students' discussion Showing multiple strategies and errors for students to explain or reason Real world problems Allowing students to tinker <p>Possible Technologies:</p> <ul style="list-style-type: none"> Desmos Dynamic Geometry Software Computer Algebra Systems Screencast software Dan Meyer videos

Note. Adapted from "A Framework for Mathematics Teachers' Evaluation of Digital Instructional Materials: Integrating Mathematics Teaching Practices With Technology Use in K-8 Classrooms," by A. Thomas & A. J. Edson, 2017, *Contemporary Issues in Technology and Teacher Education*, 19(3) (<https://citejournal.org/volume-19/issue-3-19/mathematics/a-framework-for-teachers-evaluation-of-digital-instructional-materials-integrating-mathematics-teaching-practices-with-technology-use-in-k-8-classrooms>). Copyright 2017 by A. Thomas and A. J. Edson.

The Didactic Tetrahedron

Cohen et al. (2003) presented a model, the instructional triangle, in which they described mathematics teaching practices in terms of the interactions among the teacher, students, and content. Hollebrands (2017) built upon this work by adding a fourth dimension to consider the ways in which the use of technology mediates the interactions among the teacher, students, and mathematics task (see Figure 9).

Figure 9 The Didactic Tetrahedron



From
 “The Use of Technology in the Teaching and Learning of Mathematics Framework,” by K. Hollebrands, 2016, (https://fi-courses.s3.amazonaws.com/tmt/unit_5/FrameworkMath-TechMOOC_REVISED.pdf). In the public domain.

Master, Servant, Partner, Extension of Self

As mentioned previously, Master, Servant, Partner, and Extension of Self are metaphors used to describe technology use in mathematics classrooms by both students and teachers (Goos et al., 2000). With respect to teacher use, the role of Master indicates minimal technology use in the classroom because of limited teacher knowledge. The role of Servant describes when the teacher utilizes technology in a limited way with little attempt to change the nature of mathematical tasks or students’ interactions with those tasks. When teachers use technology as a Partner, they leverage their

greater knowledge of technology to enhance the quality of students' mathematical learning. The role of Extension of Self occurs when a teacher's extensive knowledge allows them to incorporate technology in instruction in powerful yet natural ways.

Orchestration Types

Drijvers et al. (2010) reported on six orchestration types that teachers demonstrated when using a java applet in their classrooms to teach the concept of function: Technical-demo, Explain-the-screen, Link-screen-board, Discuss-the-screen, Spot-and-show, and Sherpa-at-work (see Figure 10). The first three orchestration types are considered to be more teacher-centered orchestrations, and the last three more student-centered.

Figure 10 Description of Drijvers et al.'s (2010) Orchestration Types

Orchestration Type	Description
Technical-demo	Demonstration of tool techniques by the teacher.
Explain-the-screen	Whole class explanation by the teacher, guided by what happens on the computer screen
Link-screen-board	Teacher stresses the relationship between what happens in the technological environment and how this is represented in conventional mathematics of paper, book and blackboard.
Discuss-the-screen	Whole class discussion about what happens on the computer screen.
Spot-and-show	Student reasoning is brought to the fore through the identification of and deliberate use of interesting student work in the classroom discussion.
Sherpa-at-work	A Sherpa-student uses the technology to present his or her work, or to carry out actions the teacher requests.

The PIC-RAT Framework

The PIC-RAT framework is an array with two dimensions: student use (Passive, Interactive, or Creative [PIC]) and teacher use (Replacement, Amplification, and Transformation [RAT]). Kimmons (2016) built upon the work of Hughes et al. (2006), who developed the RAT framework, which was designed to help in-service and preservice teachers make decisions about technology implementation. They drew on the work of Pea (1985) to define replacement, amplification, and transformation as the action on student learning, instructional practices or content goals. Kimmons argued that one progresses from the lower left corner (see Figure 11) or the PR section of the array toward the top right corner or CT section.

Figure 11 The PIC-RAT Framework

Student's Relationship to tech is (<u>P</u> assive, <u>I</u> nteractive, <u>C</u> reative)	CR	CA	CT
	IR	IA	IT
	PR	PA	PT
	Teacher's Use of Tech (<u>R</u>eplaces, <u>A</u>mplifies, <u>T</u>ransforms) Traditional Practice		

Note. Adapted from *K-12 Technology Integration: A Conceptual Primer* by R. Kimmons, 2016, (<https://k12techintegration.pressbooks.com/app/uploads/sites/47296/2016/09/K-12-Technology-Integration-1474477952.pdf>). Creative Commons BY-SA license.

The SAMR Model

Puentedura (2006) developed the SAMR model to provide science teachers with an approach to selecting, using, and evaluating technology (Figure 12 shows the four-step hierarchy). The Substitution level is similar to replacement in Hughes et al.'s (2006) RAT framework, in that technology is being used to replace or substitute ordinary actions. If the technology were removed, no real difference in the learning goal would occur.

At the Augmentation level technology use affords some benefits, despite the actions remaining the same. At the Modification level, technology is used to significantly redesign a task, redesign that could not have happened without the technology tool. The Redefinition level is achieved when the task itself cannot exist without the employment of that technology. Substitution and Augmentation are considered stages that enhance a teacher's practice, whereas Modification and Redefinition are considered stages that transform a teacher's practice.

Figure 12 The SAMR Model

R	REDEFINITION Technology allows for the creation of new tasks, previously made inconceivable.	Transformation
M	MODIFICATION Technology allows for significant task redesign.	
A	AUGMENTATION Technology acts as a direct substitute, with functional change.	Enhancement
S	SUBSTITUTION Technology acts as a direct substitute, with no functional change.	

Note. Adapted from “Transformation, Technology, and Education,” by R.R. Puentedura, 2006, (<http://hippasus.com/resources/tte/>). In the public domain.

The Technology Integration Planning Model

The Technology Integration Planning (TIP) Model described seven steps organized into three phases (see Figure 13) related to selecting, planning, implementing, and evaluating technology use (Roblyer & Doering, 2010). In the first phase, teachers consider learning needs and the potential that technology integration holds in addressing those needs. The goal of this phase is to clearly define the learning problem, identify a solution, consider the relative advantage of using technology, and finally, determine what resources are necessary to implement the solution.

The second phase deals with the logistics of implementation, including creating clearly defined learning goals, attending to how to evaluate those goals, and considering what instructional approach best aligns with the technology use and how to prepare students to successfully engage with the technology. The third phase focuses on assessing the technology integration. The goal of this final phase is to assess the evidence that students reached the learning goal and consider how to improve the lesson for future use.

While these six frameworks broadly address some stage of planning for technology integration and often employ a hierarchical structure, they differ in whether they attend to both use and evaluation of technology. Some are more general guidelines for planning for technology integration (e.g., The TIP Model), while others aim to consider the ways in which teachers position the technology in conjunction with learning mathematics (e.g., PIC-RAT and The SAMR Model).

Figure 13 The TIP Model

Phase 1. Analysis of Teaching/Learning Needs
Determine the relative advantage
Assess required resources and skills needed
Phase 2. Designing an Integration Framework
Decide on objectives, assessments
Design integration strategies
Prepare instructional environment
Phase 3. Post-Instruction Analysis and Revisions
Analyze lesson results, impact
Make revisions based on results

Note.

Adapted from *Integrating Educational Technology into Teaching* (5th ed.), by M.D. Roblyer & A. H. Doering, 2010, Copyright 2010 by Pearson Education.

Framing How Teachers Learn to Use Technology to Teach Mathematics

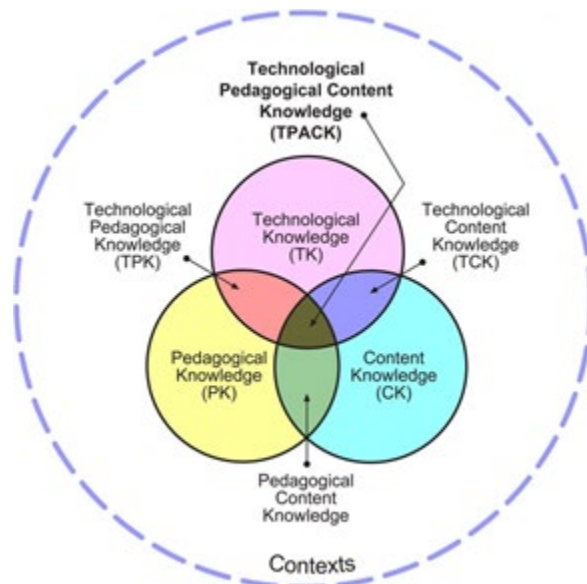
The final category of frameworks we identified were those that frame how teachers learn to use technology for teaching mathematics. These frameworks describe the specialized knowledge that teachers need for teaching mathematics with technology and how they develop that knowledge. Such frameworks are not necessarily intended for use by teachers, but they can be useful to guide the design of learning experiences (i.e., courses or professional development) or to inform research that includes identifying or developing teachers' knowledge for teaching mathematics with technology. The two frameworks that fell into this category – TPACK and Play, Use, Recommend, Incorporate, and Assess (PURIA) – are described in the following section.

Technological Pedagogical and Content Knowledge

The TPACK framework describes the specialized knowledge needed to teach with technology (Mishra & Koehler, 2006). TPACK is an expansion of Shulman's (1986) pedagogical content knowledge (PCK), which described the intersection of knowledge of content and knowledge of teaching. TPACK, as described by Mishra and Koehler, asserts that effective teachers rely on the integration of multiple domains of knowledge: deep understanding of the content (e.g., mathematics), deep understanding of pedagogy, and a deep understanding of technology (see Figure 14).

In the case of mathematics, Niess (2005) and Niess et al. (2009) adapted Grossman's (1989, 1990) proposed components of PCK to take technology into consideration. Neiss et al.'s (2009) adaptation included using the elements of the framework to outline the outcomes of TPACK knowledge:

Figure 14 The TPACK Framework



Note. From "TPACK Explained," by M. Koehler, 2012 (tpack.org). In the public domain.

(1) an overarching conception about the purposes for incorporating technology in teaching mathematics; (2) knowledge of students' understandings, thinking, and learning of mathematics with technology; (3) knowledge of curriculum and curricular materials that integrate technology in learning and teaching mathematics; (4) knowledge of instructional strategies and representations for teaching and learning mathematics with technologies. (p. 8)

The PURIA Model

The PURIA model (Beaudin & Bowers, 1997) describes developmental phases teachers go through to become confident teachers of mathematics with technology. Originally conceived as a way to describe how teachers learn to integrate computer algebra systems (CAS) into their teaching, the model was later expanded by Zbiek and Hollebrands (2008) to include all mathematics action tools. Each stage is described in Figure 15.

In this model, teachers develop through using technology in one mode until they are comfortable enough to go on to the next mode of use. Stages can overlap and are traversed at different rates depending on the aptitude of the user. Zbiek and Hollebrands (2008) noted that PURIA is useful for understanding "not only how teachers' understandings grow but also why teachers' personal technology use and classroom practice may differ greatly" (p. 334).

Figure 15 The PURIA Model

PURIA mode	Description of Activity	Nature of Activity
Play	User plays with the technology	No clear mathematical purpose in playing with the software.
Use	Technology is used as a personal tool	Does mathematics of own design. May use it as a learner of mathematics, but not in a formal classroom setting or with students.
Recommend	Suggests to others that they investigate / use the technology	Recommends to a peer, individual or small group of students. Still not in a formal classroom setting or integrated part of teaching.
Incorporate	Use the technology as part of classroom teaching	Starts using the technology in formal teaching environment. Incorporates technology into lessons to varying degrees. (Positive experiences in Play and Recommend are required to get to this stage.)
Assess	Use the technology to assess students	Uses the technology to assess what students are learning in terms of technology and mathematics.

Note. Adapted from “A Research-Informed View of the Process of Incorporating Mathematics Teachers’ Processes for Making Sense of Students’ Work with Technology,” by R. M. Zbiek and K. Hollebrands, 2008. *Research on Technology and the Teaching and Learning of Mathematics 1*, 287-344. Copyright 2008 by National Council of Teachers of Mathematics.

In summary, the two frameworks in this category are different, but complimentary. TPACK describes knowledge needed to teach mathematics with technology and PURIA is a process one might go through to develop that knowledge.

Implications for Mathematics Teacher Educators

The findings from this study provide an overview of the ways the work of MTEs related to the preparation of teachers to teach mathematics with technology both is and is not being framed. The MTEs who responded to this item on the survey all listed frameworks well aligned with the work of preparing teachers to teach with technology. While some frameworks were elicited that were not framing particular phenomena (e.g., Ohio Learning Standards for Technology), they were all in some way related to teaching and learning with technology. Focusing on the 17 frameworks that were framing a phenomenon, and hence were included in this study, approximately three quarters of them were derived from mathematics specific contexts. However, the two most commonly elicited frameworks, TPACK (21%) and the SAMR model (12%), were not. These results lead us to reflect on not only what is being framed by this collection of frameworks, but also what might be missing.

What Is Being Framed?

Our analysis of the 17 frameworks was guided by identifying the phenomena related to teaching mathematics with technology that were being framed. From this lens, we identified four categories:

1. How students use and learn with technology.
2. The design and evaluation of technology tools and tasks.
3. How teachers use technology.
4. How teachers learn to use technology.

Two to seven frameworks fit in each category (see Figure 16), with a variety coming from mathematics-specific contexts and nonmathematics-specific contexts across the four.

Figure 16 List of Frameworks by Category

How students use and learn with technology
<ul style="list-style-type: none"> • Amplifier & Reorganizer (Pea, 1985) • Activity Theory (Engeström et al., 1999; Nardi, 1996) • Master, Partner, Servant, Extension-of-self (Goos et al., 2000) • Virtual Cultural Theory (Shaffer & Kaput, 1998) • White Box/ Black Box (Buchberger, 1990)
How teachers use technology
<ul style="list-style-type: none"> • Orchestration Types (Drijvers et al., 2010) • Master, Partner, Servant, Extension-of-self (Goos et al., 2000) • The PIC-RAT Framework (Kimmons, 2016) • The Didactic Tetrahedron (Hollebrands, 2017) • The SAMR Model (Puentedura, 2006) • The Technology Integration Planning (TIP) Model (Roblyer & Doering, 2010)
The design and evaluation of technology tools and tasks
<ul style="list-style-type: none"> • Framework for Evaluating Digital Instructional Materials (DIM) (Thomas & Edson, 2019) • Dynamic Geometry Task Analysis (Trocki, 2014) • Interactive Geometry Software (IGS) Framework (Sherman & Cayton, 2015) • Pedagogical, Mathematical, and Cognitive fidelity (Dick, 2008) • Design Principles for Pre-Constructed Dynamic Geometry Sketches (Sinclair, 2003)
How teachers learn to use technology
<ul style="list-style-type: none"> • Technological Pedagogical and Content Knowledge (Mishra & Kohler, 2006) • The PURIA Model (Zbiek & Hollebrands, 2008)

Approximately half of the frameworks in the “How students use and learn with technology” and “How teachers use technology” categories did not originate in a mathematics-specific context. Those that did not originate in mathematics contexts often originated in general education contexts instead. In contrast, all of the frameworks related to the design and evaluation of technology-based tasks were developed in a mathematics-specific context.

This result makes sense, as these frameworks are related to mathematics-specific technology tools. In fact, some of the frameworks in this category were related to very specific tools. For example, three of the six are specific to dynamic geometry software. It is worth considering the benefit of both more and less specific frameworks in this category. As mathematics-specific tools have broadened to include more than dynamic geometry spaces (e.g., Desmos Activity Builder), having frameworks to guide the

selection or adaptation of their use would be helpful to teachers. That said, frameworks specific to particular software capabilities might be more helpful in guiding the work of curriculum designers working to incorporate such technology tools in meaningful ways.

Reflecting on the frameworks that MTEs are using that are not mathematics specific, we wonder what benefit there may be in adapting them for mathematics specific contexts. For example, TPACK started as a framework to describe knowledge to teach with technology broadly (Mishra & Koehler, 2006). Neiss (2005) and Neiss et al. (2009) then adapted it to frame knowledge for teaching science and mathematics specifically.

Others have since adapted it to be even more specific, focusing on knowledge to teach statistics (Lee & Hollebrands, 2011) and functions (Lovett et al., 2020) with technology. Such specified frameworks have proven useful for the development of both curricula and research activities related to the teaching of mathematics and statistics with technology (e.g., Ayieko et al., 2019; Lee & Hollebrands, 2008; Lee & Hollebrands, 2011; Lovett, et al., 2020; Niess, et al., 2009).

That all but one of the frameworks in this study has been peer reviewed is encouraging. Our work as MTEs should be guided by research (AMTE, 2017). Knowing a framework has gone through a rigorous review process provides MTEs' confidence in its usefulness and insight into ways it can guide various aspects of our work.

The only framework submitted that has not gone through the peer review process is The SAMR Model. In fact, this framework has been criticized for having not gone through peer review, and its shortcomings are well documented (Hamilton et al., 2016). Yet both our findings here and those of Hamilton et al. (2016) indicate that it is widely used across a variety of settings.

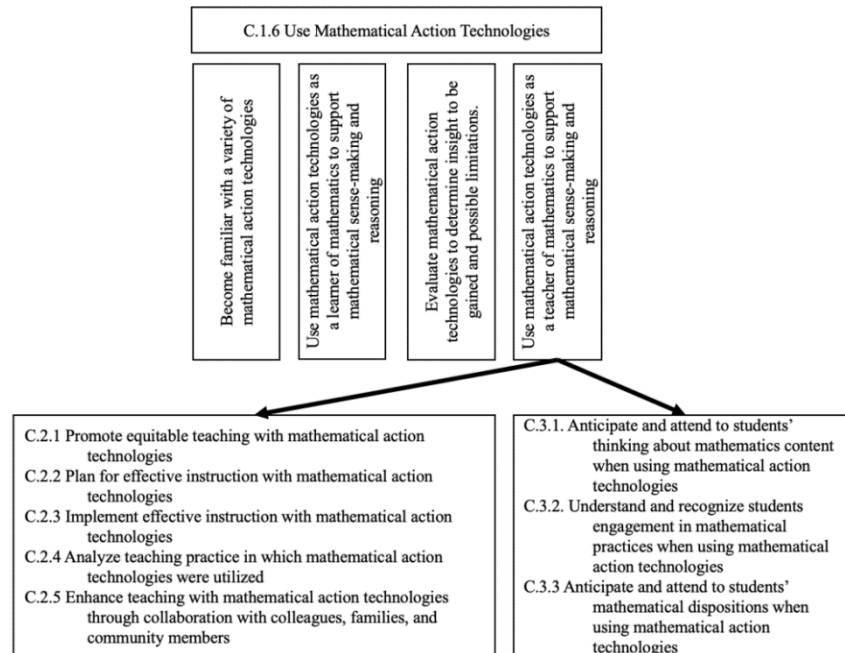
This wide use suggests something about the framework is useful to MTEs. It may simply be that the SAMR model has appeared in more visible venues, so more people are aware of it, or maybe it offers something that other frameworks do not. For example, the primary source for the SAMR model is a presentation created by the head of an educational consulting firm (Hippasus). This presentation has been disseminated widely through paid professional development sessions at both the school and district level (Puentedura, 2006), possibly making it more visible to teachers and professional developers than are frameworks disseminated through peer reviewed journals. Determining the reasons for its popularity would be fruitful, as such findings could guide the ways in which frameworks intended for use with teachers are communicated.

What Is Not Being Framed?

Given our findings related to what is being framed, we find it prudent to step back and consider phenomena related to MTEs' work to prepare PSMTs to teach mathematics with technology that is *not* being framed. To do so, we compared the resulting collection of 17 frameworks with the

aspirational goals outlined in the AMTE (2017) SPTM related to teaching with technology (see Figure 17).

Figure 17 Framing the Goals for Preparing Teachers to Teach Mathematics With Technology



Note. Adapted from “Addressing the SPMTs: Critical conversations about preparing mathematics teachers to utilize technology in their instruction” by A.W. McCulloch, K.R. Leatham, J.N. Lovett, N.G. Bailey, C. Cayton, and S. Reed, 2019. *PME-NA Proceedings*. (<http://www.pmena.org/proceedings/>). In the public domain.

One of the first things we notice across all categories was that equity issues are not explicitly addressed in any of the frameworks in this sample. While the language in some of the frameworks seems to provide space for considering issues of equity, none specifically frame issues related to equity to inform teaching and learning with technology. For example, when considering the framing of work related to designing and evaluating tasks, not only would you expect attention to the ways the task connects to students’ personal assets broadly, but also specifically related to the technology (e.g., is the task accessible based on prior technology experiences? AMTE, 2017).

Furthermore, issues of equity should inform decisions about technology use as it relates to access, yet none of the frameworks in this sample address how decisions are made regarding when and what to use. In a study of 21 technology-using early career mathematics teachers, McCulloch et al. (2018) found regularity among the criteria teachers consider when deciding whether to incorporate technology into a lesson and, when deemed appropriate, for deciding which technology tool to use. Many of the criteria noted were related to issues of access (e.g., Does the tool work on a variety of platforms? Does it have to be downloaded? Do students need to have an account to use it?). These findings suggest that a

framework to guide such teacher decisions would be useful. (For an example of an equity-related framework for technology use in the classroom, see Damarin, 2000.)

Another area of work noticeably missing among this group of frameworks is attention to the ways that eliciting and using student thinking differs when students are using mathematics action technologies compared to when they are not. Many researchers have observed that the ways students interact with a technology tool, especially a dynamic mathematics tool, can provide insight to student thinking (e.g., Azarello et al., 2002; Doerr & Zangor, 2000; McCulloch, 2011; Wilson et al., 2011). These observations, coupled with recent research that points to the importance of PSMTs' developing the skill of professional noticing of students' mathematical thinking (Jacobs & Spangler, 2017), suggest a need for framing the noticing of student mathematical thinking when using technology.

We briefly mention one other noticeably missing intersection of technology and mathematics education missing in this collection of frameworks: classroom mathematics discourse. A great deal of attention has been paid in the past few decades to the critical role discourse can play in the learning of mathematics (Herbel-Eisenmann et al., 2017) and, with that emphasis, explorations into the ways technology can influence that discourse (see Leatham & Barton, 2017). Promising directions in this area include work by Anderson-Pence (2017) and adapting existing mathematics discourse frameworks, such as Adler and Ronda (2015).

Conclusion

While our findings provide a helpful snapshot of the ways MTEs are framing the work related to teaching PSMTs to teach with technology, this study has limitations. The sample of frameworks we examined were not drawn systematically from the complete set of frameworks available for such work, rather they were offered by participants who completed a particular item on the survey: "Which frameworks related to technology (if any) do you share explicitly with students in these classes?" The wording of this item limited the responses to frameworks MTEs use with their PSMTs, thus, not providing a full picture of the frameworks that might be informing their work as a whole. For example, the category of framing how teachers learn to use technology only has two responses, likely due to this wording, as such frameworks guide the work of MTEs but might not be shared explicitly with PSMTs.

Our analysis of this collection of frameworks was guided by our research questions. As such, we focused on the phenomenon each was framing. However, if we had asked a different question, we might have categorized them differently. For example, one might think about whose work each framework was intended to inform (e.g., MTEs designing learning experiences for PSTs or PK-12 teachers designing instruction for their students).

Another possibility would be to examine the frameworks in reference to whether they are descriptive (models of what happened) versus prescriptive/aspirational (models of what should happen). This possibility

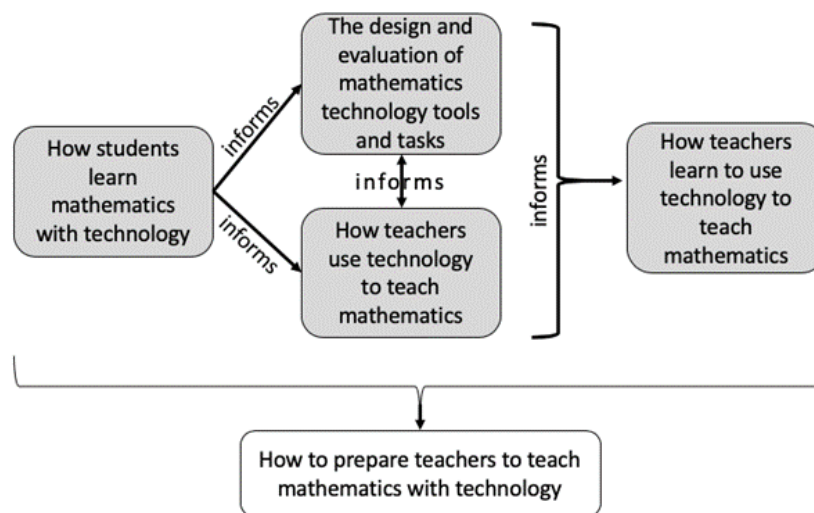
brings up a critical point related to our decision to write our summaries based on the original use of a given framework.

Our summaries did not take into account how these frameworks might be used beyond their originally stated intent. For example, within the category of “Framing how teachers use technology”, Drijvers et al. (2010) described six technology orchestration types that teachers were observed using during instruction, which reports on models of what happened with technology use during instruction. However, the potential exists for MTEs to use this framework with PSMTs to educate them on which orchestration types should be the focus of teachers’ technology use during instruction.

In any of these scenarios, we hypothesize that themes may not be mutually exclusive. In fact, within the themes from our results Master, Partner, Servant, Extension of Self (Goos et al., 2000) was highlighted in both “How students use and learn with technology” and “How teachers use technology.” We would encourage fellow researchers, not only to think about different ways of examining the frameworks highlighted in this paper, but also to consider potential uses beyond the original purpose.

Furthermore, other areas of MTEs’ work related to preparing PSMTs to teach with technology likely exist that are neither captured by this study nor apparent in the mathematics teacher education literature. For example, we do not have frameworks to guide the complex interplay among providing PSMTs experience learning with technology, designing and evaluating tasks and instruction, and actually teaching with technology. In other words, given frameworks that frame student learning, teaching, and learning to teach mathematics with technology, what we do not seem to have yet are those for framing the teaching of mathematics teachers. Such a framework would likely draw heavily on frameworks like those discussed in this paper but would need to frame the work of MTEs (see Figure 18).

Figure 18 Framing the Work of MTEs to Prepare Teachers to Teach Mathematics With Technology



As our collective work toward preparing teachers to teach mathematics with technology continues to progress, it is important to consider the ways we are “using tools and frameworks grounded in research to develop core pedagogical practices” (AMTE, 2017, p. 35). In this study we utilized the frameworks identified by a survey of all accredited U.S. secondary mathematics teacher preparation programs to begin a conversation about the frameworks specific to this work that have emerged, as well as those that might still be needed. The findings from this study indicate a robust collection of frameworks that MTEs use in their instruction related to ways students’ use and learn with technology, ways teachers use technology, and selecting, evaluating, and adapting technology based tasks. However, fewer frameworks are related to ways teachers learn to use technology in their instruction (although the small numbers of such frameworks in our results may have been due to the focus of the survey prompt).

As we noted at the beginning of this paper, comparing and contrasting frameworks to identify overlaps and gaps are important tasks (Lester, 2005). We began that work here, noting overlaps, like the multiple frameworks specific to the design and evaluation of dynamic geometry tasks, and gaps, like the lack of explicit attention in these frameworks to issues of access and equity.

Future work focused on reconciling differences among overlapping frameworks as well as addressing identified gaps is necessary if these frameworks are going to guide work with PSMTs toward achieving the AMTE SPTMs. In addition, the findings here suggest that we are poised to consider how the frameworks in the four categories identified in this study could together inform an overarching framework (see Figure 18) for the work of MTEs in preparing secondary mathematics teachers to teach with technology.

Acknowledgements

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