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Why and How Secondary Mathematics Teachers Implement Virtual Manipulatives

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Although teachers are expected to teach with technology, they often are not prepared or supported to do so (Albion, Tondeur, Forkosh-Baruch, & Peeraer, 2015), a critical issue in mathematics education (Wilson, 2008). The study described in this article investigated why and how secondary mathematics teachers implemented virtual manipulative (VM) tasks during and after participating in a professional development (PD) opportunity aimed at teaching with VMs. Findings indicate that teachers used VM tasks due to instructional benefits, for example, supporting students' developing understanding and differentiation. Additionally, they used VMs and tasks due to the support they received from tools introduced during the PD. In this study, teachers primarily used VM tasks to support students' developing understanding, to provide in-the-moment feedback, and as a reteaching tool. Mediating factors, such as student needs, curriculum, time, tool limitations, and so forth, influenced why and how teachers chose to use a particular VM.

In today's classrooms, teachers are expected to integrate technology as a means to enhance student understanding and engagement. Depending on the content area, the *technology tool* can take on many forms (e.g., interactive white boards, tablets, computers, games, software, virtual manipulatives, etc.) and roles (see Cullen, Hertel, & Nickels, 2020). Yet, teachers often report that they are not prepared to integrate technology in an effective and innovative manner (Albion, Tondeur, Forkosh-Baruch, & Peeraer, 2015).

Over the past 30 years, virtual manipulatives (VMs) are one technology tool that teachers have been encouraged to implement to enhance student learning (Association of Mathematics Teacher Educators [AMTE], 2017; National Council of Teachers of Mathematics [NCTM], 2000). A VM is an "interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge" (Moyer-Packenham & Bolyard, 2016, p. 13).

As with any technology tool, the potential for VMs to support student learning lies not within the tool itself but in how students engage with the VM and the relevant mathematics (Moyer-Packenham & Bolyard, 2016). Therefore, teachers must be prepared to create opportunities that promote quality engagement with VMs, thus presenting opportunities for students to develop conceptual understanding.

Unfortunately, most of the studies investigating teachers' use of VMs occur at the prekindergarten through Grade 6 levels, with minimal studies conducted at the middle school, high school, and above levels (Moyer-Packenham & Westenskow, 2013). Therefore, this article reports the findings of a study on a professional development (PD) opportunity for secondary mathematics teachers aimed at supporting their efforts to teach with VMs. (See http://bit.ly/VirtManips for an annotated list of VM collections and http://bit.ly/VMActivities for a repository of VM tasks used during the PD.)

Virtual Manipulatives

VMs are commonly categorized into five environments: single representation, multirepresentations, tutorial, gaming, and simulation (Moyer-Packenham & Bolyard, 2016). The *single-representation* VM "environment contains an interactive pictorial/visual representation (i.e., image) of a dynamic mathematical object and is not accompanied by numerical or text information" (p. 14).

Typically, this environment requires teachers to design tasks to focus students' attention on the relevant mathematical ideas and learning goals. The *multi-representation* "environment contains the interactive visual representation (i.e., image) of the dynamic mathematical object and is accompanied by numerical and, sometimes, text information" (Moyer-Packenham & Bolyard, 2016, p. 15). Typically, the numeric information is linked simultaneously with the visual representation, thus promoting students to make connections and see patterns more easily.

The *tutorial* "environment contains the interactive visual representation (i.e., image) of the dynamic mathematical object and is accompanied by numerical and text information in a format that guides the user through a tutorial of the mathematical procedures and processes being presented" (Moyer-Packenham & Bolyard, 2016, p. 17). The tutoring and guiding characteristics of this environment distinguish it from the multirepresentation environment.

In the *gaming* environment, the VM is "embedded in a format that allows the user to play a game with the object to reach goals that are reflected in the gameplay" (Moyer-Packenham & Bolyard, 2016, p. 18). Finally, in the *simulation* environment the VM is an interactive image "of the dynamic mathematical object" and other representations (e.g., text and numeric) embedded in a manner that enables "the user to run a simulation intended to represent or draw attention to embedded mathematics concepts" (p. 20).

Teachers in the PD described here interacted with VMs from all five environments; however, VMs were primarily from the single-representation, multirepresentation, and tutoring environments. Appendix A contains examples of VMs from the different environments that teachers interacted with during the PD.

Moyer-Packenham and Westenskow's (2013) meta-analysis of studies investigating the effects on student achievement when VMs are used identified five interrelated categories of affordances of VMs:

> *focused constraint* (i.e., VMs focus and constrain students' attention on mathematical objects and processes), *creative variation* (i.e., VMs encourage creativity and increase the variety of students' solutions), *simultaneous linking* (i.e., VMs simultaneously link representations with each other and with students' actions), efficient precision (i.e., VMs contain precise representations allowing accurate and efficient use), and motivation (i.e., VMs motivate students to persist at mathematical tasks). (p. 35)

Amongst the affordances, the interaction between the dynamic object, the learner, and the mathematics is what determines the actual affordance of the VM for student learning (Moyer-Packenham & Westenskow, 2013). Both Tucker, Moyer-Packenham, Westenskow, and Jordan's (2016) and Moyer-Packenham and Westenskow's (2016) more recent meta-analysis found that creative variation was the least frequent affordance category identified by evidence as contributing to student learning.

Beyond these affordances, additional reasons exist for promoting teachers' use of VMs in the classroom. For example, studies indicate that VMs can be used as a tool to support students' learning by providing opportunities for immediate feedback (Edwards Johnson, Campert, & Zuidema, 2012). In today's dynamic classrooms, VMs can be used to support teachers' differentiation efforts (Bouck, Flanagan, & Bouck, 2015; Shin et al., 2017) to support students' emerging understanding, as well as a means for challenging (or encouraging) students to explore a mathematical idea from a different perspective. Bouck and colleagues (2015) also suggested that VMs benefit students with learning disabilities in both their performance, their confidence, and possibly, their development of conceptual understanding. Finally, VMs can provide equal access for students to learn content by reducing effects of students' demographics (socioeconomic status and English language learner status) as predictors of achievement (Moyer-Packenham et al., 2014).

Providing access to technology tools is "not enough" in supporting teachers to teach with the tools; rather teachers need to "come to know the appropriate and constructive uses of technology" (Wilson, 2008, p. 415). Our PD aimed to support teachers implementing VMs and tasks in appropriate and constructive ways. Since minimal studies have been done investigating secondary mathematics teachers' use of VMs, this study aimed to extend the findings from studies at the elementary levels investigating teachers' implementation of VMs to the secondary levels. Specifically, this study explored the following research question: Why and how do secondary mathematics teachers report implementing virtual manipulatives while participating in a focused professional development opportunity?

Theoretical Background

Rather than studying teachers' implementation efforts in isolation, teachers' implementation of VM tasks are considered to be mediated by their mathematical goals for the task, the tools available related to their implementation (i.e., technology and otherwise), and the students they teach (Zbiek, Heid, Blume, & Dick, 2007). A VM task refers to a VM and all accompanying instructional materials (e.g., prompts and directions) whether on screen or in printed form. The VM task could include more than one task (Sinclair, 2003) focused on investigating a particular concept (e.g., through alternative exploration paths), but it may include only one task.

To understand why and how teachers implemented VMs, the following study drew from the third strand of activity theory (Engeström, 1987, 1999; Nardi, 1996). Activity theory takes a multidimensional approach to investigating activities in which people are engaged and acknowledges that activities are mediated by the context of the activity, the subjects' experiences, how they use tools and how tools are transformed through activity and so forth. Engeström (1987, 1999) discuss how this strand draws and differs from the Russian strand of activity theory originating with Vygotsky, Leont'ev, and Luria.

An activity consists of a subject, object, and actions. In this study, the teachers engaged in the PD are the *subject,* as they are the learners. The *object,* which motivates the activity and gives the activity specific direction are the teachers' implementations of VMs and VM tasks.

Teachers' practices related to planning and preparing to implement VMs and tasks constitute the *actions* (the goal-directed processes undertaken by the subject to achieve the object) of an activity system in this study. "Actions are chains of operations" (Jonassen & Rohrer-Murphy, 1999, p. 63) and require conscious effort. Through repeated practice, actions can become operationalized (subconscious). Examples of actions include critiquing a VM task, process of developing an instructional guide to accompany a VM, and so forth. Other aspects of an activity system (see Figure 1) include the tools/mediating artifacts (the VMs and tasks, the task analysis framework), rules (curriculum, instructional style), community (their students and other teachers in the school), and division of labor (do teachers work collaboratively or primarily individually). The focus of this study was on why and how teachers implemented VMs and tasks (the object).

Methods

To address the gap in literature about secondary mathematics teachers' use of VMs and tasks, the aim of this study was to investigate why and how secondary mathematics teachers reported they implemented VMs and tasks while participating in a PD opportunity. Teachers' conversations and responses during the PD sessions, as well as interviews with some of the participants, were used to investigate teachers' reported implementation of VM tasks.

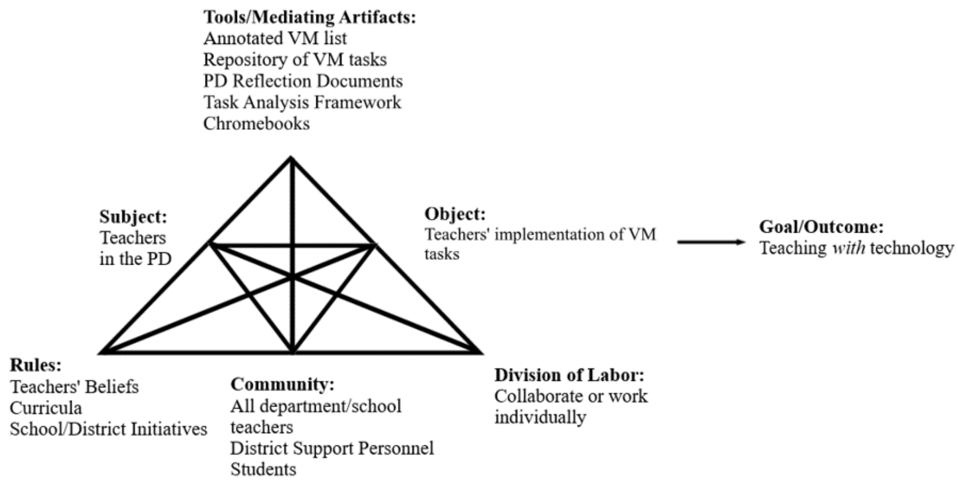

Figure 1. *Structure of an activity system for this PD.*

Professional Development Opportunity

The design of the PD was based on five components of effective PD (e.g., Desimone, Porter, Garet, Yoon, & Birman, 2002). That is to say, *content focus* (grounded in the teachers' mathematics curriculum), *active learning* (teachers engaged with VMs and tasks related to their identified learning goals), and *duration* (at least 20 contact hours spread across 6 months). The PD was coherent with the district initiatives regarding teacher and student technology use and promoted collaboration amongst participants through collective participation (teaching pairs from the same school).

In phase I, teachers reflected on the role of technology in the classroom and their current technology use. Additionally, tools (guiding questions and a task analysis framework) were introduced to support teachers' integration efforts and knowledge growth (see Appendix B; Reiten, 2018). The task analysis framework drew from work by Trocki (2014) and Sinclair (2003) with dynamic geometry software tasks and was intended to help teachers critique and develop tasks aimed at promoting students' development of conceptual understanding of mathematics through reflection and communication (Hiebert et al., 1997), as well as through using and connecting mathematical representations (NCTM, 2014).

During this phase, teachers completed, critiqued, and compared three VMs/VM tasks (i.e., Modeling and Solving Two-Step Equations from ExploreLearning, https://www.explorelearning.com/index.cfm?method=cResource.dspDetail&ResourceID=226; Algebra Tiles from Illuminations, https://www.nctm.org/Classroom-Resources/Illuminations/Interactives/Algebra-Tiles/; and Virtual Algebra Tiles from Michigan Virtual University, http://media.mivu.org/mvu_pd/a4a/homework/applets_two_step.html) for solving two-step equations. As teachers progressed from the first phase through the third phase, they took on more responsibility for finding and selecting the VMs and tasks.

Beginning with the second PD session, teachers recorded their responses to the guiding questions and how they were using the task analysis framework using

Google Docs. The Google Docs encouraged teachers to use the task analysis framework to critique VMs and tasks. Additionally, the Google Docs included a prompt that encouraged teachers to reapply the task analysis framework after they made or thought about potential modifications/adaptions to the VMs and tasks that they were exploring. (Appendix C contains an example of a Google Doc from the latter portion of the PD).

Phases II and III focused on teachers' learning goals and refining their instructional practices related to implementing VM tasks. In Phase II teachers were given VM tasks to critique, whereas in Phase III teachers found the tasks to critique and modify. During the PD sessions in these two phases, teachers spent the first part of each session completing and critiquing VM tasks related to their identified learning goals.

For the remaining part of the PD session, teachers selected a VM or task and then spent time preparing the task to use with their students (e.g., modifying or creating instructional guides). Due to the attention given to individual teacher's growth, some teachers were at Phase II while other teachers were at Phase III. Teachers used the guiding questions and task analysis framework throughout the three phases of the PD as they critiqued and modified/developed instructional guides to accompany VMs. Therefore, teachers' use of these tools constituted some of the actions investigated during the study that supported teachers' implementations of VM tasks (the object) of an activity system.

Participants

Fourteen teachers in a suburban district participated in a PD opportunity aimed at fostering their use of VMs and tasks. Middle school teachers were originally encouraged to participate due to all middle school students having Chromebooks. Ten middle school teachers, three high school teachers, and one fifth-grade teacher comprised the participants. Table 1 contains additional information about the participants. In the remainder of this article, "teacher" or "teacher(s)" refers to the teachers who participated in the PD and "students" refers to the teachers' students.

Data Sources and Analysis

Data collected throughout the PD included initial background and final surveys (both of which were online), video and audio recordings of the PD sessions, as well as teachers' responses to Google Docs (Appendix C) containing a set of guiding questions and encouragement to use the task analysis framework. The task analysis framework and guiding questions are in Appendix B.

Four teachers volunteered to participate in audiorecorded semistructured interviews at the beginning and end of the PD. Additionally, these teachers and an additional teacher volunteered for brief interviews before and after implementing VM tasks with their students. Transcripts were created from all recordings.

Table 1
Information About Participants in the Professional Development Opportunity

Pseudonym	School	Grade/Course Taught
Kelly	Prospect ES	5th
Josh [a]	Wetlands MS	6th
Mark [a]	Wetlands MS	6th
Tracy [a]	Wetlands MS	6th/7th Interventionist
Curt	Plains MS	7th
Karen	Plains MS	7th
Randy	Plains & Summit MS	6th/7th/8th Interventionist
Erin	Summit MS	8th
Stan	Summit MS	8th
Mari	Summit MS	8th
Pam	Summit MS	8th Interventionist
Daron [a]	HS	Geometry, AP-Stats
Jake [a]	HS	AP-Calculus, Pre-Calculus, Consumer Math
Mike	HS	AP-Calculus, Pre-Calculus, Transition to College Math

[a] Teachers who volunteered for interviews during the PD.

An open coding approach (Saldaña, 2013) using NVivo was initially applied to the transcripts and teachers' responses on the Google Docs to identify themes regarding why and how teachers implemented VMs and tasks. Transcripts and Google Docs were initially coded based on components of the activity system. For example, discussions about students and fellow teachers were coded as *community*; statements about implementing VMs and tasks were coded as *object*. Then, transcripts and Google Docs were coded again in their entirety.

Drawing from the literature, some codes investigating why teachers implemented VM tasks included differentiation, feedback, exploration, and engagement. Some emergent codes investigating how teachers reported implementing VM tasks included supplement curriculum, solidify understanding, and the types of modifications. These emergent codes were used rather than Moyer-Packenham, Salkind, and Bolyard's (2008) seven categories [a] articulating how K-8 teachers used mathematics tools to better capture how teachers in this study articulated using VMs and tasks.

Additional rounds of analysis occurred using the constant comparative method (Glaser & Strauss, 1967) to refine code definitions based on themes between data within categories and then focused coding proceeded. Connections in the data between components of the activity system (tools, community, and division of labor) gave insight into how teachers implemented VM tasks (the object).

All data sources were coded using the same codes (see Tables 3, 4 and 5 for codes). A data excerpt consisted of at least a complete sentence in each data source. To minimize the potential of segmenting the data too much, at times a datum excerpt contained multiple sentences if all the sentences were related to the same code (e.g., see the example for *Practice* in Table 2). When more than one code applied to a datum excerpt (see Daron's response to a question on the final survey in Table 4), the excerpt was coded for each relevant code.

Findings presented are based only on the data collected during the PD sessions, interviews, and online surveys. Due to other afterschool commitments, one eighth-grade teacher stopped participating in the PD after the first two sessions. Teachers' implementation of VMs and tasks between PD sessions is not included in the presented findings unless teachers specifically talked about their implementation efforts during a PD session, interview, or survey. Therefore, it is possible that teachers implemented more VMs and tasks than indicated in the data.

Findings

Teachers' appropriation of tools (i.e., guiding questions, task analysis framework, annotated VM list, and folder of tasks) introduced during the PD supported their implementation efforts (Reiten, 2018). Teachers' responses on the initial background survey indicated that only three out of 14 teachers felt comfortable integrating VMs before the PD. However, at the conclusion of the PD, 10 teachers stated that they regularly implemented VMs and tasks and felt comfortable doing so.

The presented findings and discussion highlight teachers' conversations and responses to exemplify why and how they implemented VM tasks with their students. Although the goal of the PD was to support teachers in implementing VM tasks, it was not known what influence the PD and support tools may have on teachers' practices.

Why Teachers Implemented VM Tasks

Two themes emerged from the data regarding why teachers implemented VMs and tasks. The first theme relates to the various instructional benefits teachers identified that supported their students' emerging understanding as to why they chose to implement VMs and tasks – meaning that teachers' actions related to preparing to implement VM tasks were influenced by how they thought their students' (part of their community) emerging understanding was supported. Additionally, teachers identified implementing VMs and tasks (object) because the VM tasks contained instructional guides and questions that further supported their learning goals (rules). Both themes are discussed in more detail in the following subsections.

Instructional benefit. Teachers implemented VMs and tasks because they thought their students benefited or their learning was enhanced due to students' engagement with the VM tasks. The following subsection discusses the benefits teachers identified and then provides examples of how Tracy (a middle school teacher) and Daron (a high school teacher) thought their students benefited from interacting with VMs and tasks.

Table 2 highlights the different types of potential benefits teachers described related to why they used VMs and tasks. The number of data excerpts related to each category are included in the table for descriptive purposes only, as multiple data sources may have contributed to the number of data excerpts. As displayed in the table, teachers identified the visual nature of VMs, opportunities for student exploration, and assisting in differentiating instruction as the most common benefits to instruction and student understanding. Teachers also cited students benefiting from the use of VMs and tasks: Students could offload computational aspects of a task, receive feedback, practice problems (including extending beyond the VM), and connect multiple representations. The VM also encouraged student engagement in the lesson and learning goal.

Teachers identified other student benefits as to why they chose to use VM tasks that are not included in the table due to the low occurrence of these benefits being mentioned: communication (promoted students' discussion about what they were doing and learning), conceptual understanding (promoted opportunities to develop students' conceptual understanding), concrete (potential for making an abstract concept more concrete), make and test conjectures (provides opportunities for students to make and test conjectures), and using technology (using technology is a benefit in and of itself). These benefits occurred six or fewer times in the dataset. Although not included in the table, these benefits are still noteworthy and may contribute to why other teachers consider integrating VM tasks in their instruction.

During a final interview, Tracy shared her thoughts as to why she chose to implement VM tasks:

> It piques their interest, in all of it. All of a sudden we'll [i.e., the students] TRY because we're not writing on pencil and paper. Like the one that we did with the area and the perimeter when they're making the gardens [i.e., Fido's Flower Bed [b] from ExploreLearning] all of a sudden, "*OH! A man walked by.*" Or "*Oh, did you see that dog?!*" You know, I mean … they get excited about it and they can compare [with each other]. … They had a little liberty as to HOW they could make it. … So, instead of drawing it, they can click. So I think it piques their interest a little bit, too. [Capital letters used to signify emphasis in speech; italics used to signify change in tone.]

Table 2
Categories, Definitions, and Examples of Teacher Identified Student Benefits When Using VMs and Tasks

Definition	Number of Data Excerpts [a]	Example
Connect Multiple Representations: opportunities for students to connect multiple representations	8	**Mark (DecGoogleDoc):** "The connection between the area model, the measurements, and the formula for area and perimeter of rectangles makes this a meaningful lesson."
Differentiation: opportunities for personalized learning by differentiating instruction based on student needs.	16	**Josh (MarchPDSummary):** Josh shares his thoughts on the assessment questions at the end (i.e., Activity B) one of the questions he would get about 50% of the students and Q5, he will get about 2% of the students, which would be perfect. Josh says it would push the students in different directions.
Engagement: promotes student engagement (e.g., through motivation and/or piquing their interest).	9	**Mark (FinalSurvey):** "A larger number of students are engaged. All students have an opportunity to interact with the problem(s) they are attempting to solve."
Exploration: promotes opportunities for students to explore concepts through manipulation, interaction and so forth.	27	**Jake (FinalSurvey):** "One advantage is, kids, you-they can manipulate. I mean, they can move things around, they can change parameters, they can change…"
Feedback: provides opportunities for students to receive some type of feedback.	11	**Jake (FinalSurvey):** "…some of them they can get feedback instantly or very close to instantly. So-it allows, when you are doing something, if they are getting that feedback, that you don't have to be at, 20 different 20 different people or 20 different desks or whatever."
Modeling: provides opportunities for students to use or develop models.	8	**Kelly (NovGoogleDoc):** "Being able to model division situations and having a better understanding of how remainders happen and what they mean."
Offload: due to built in features (e.g., calculations, drawing graphs) of the VM, students can focus on the content rather than	12	**Pam (MarchGoogleDoc):** "The students would benefit from this because it's less time-consuming than creating the intersecting graphs with a pencil and paper.

Definition	Number of Data Excerpts [a]	Example
the *menial* or *tedious* components of a task.		They would be able to practice with more solid examples."
Practice: provided opportunities for students to practice problems.	11	**Tracy (OctPD):** "It's is a-it's got a lot of sequential steps. I mean, it starts with the very very basic of just learning how to build them. And then gets you going, like you can keep clicking to the next level to get more difficult, more difficult, more difficult. So, umm, it's great practice for kids who, just don't even understand what Algebra tiles are."
Visual: provides a visual of the concept being investigated.	30	**Mari (DecGoogleDoc):** "Students would benefit with the sliding aspect of the gizmo. Seeing the function change as the y-intercept changes or as the growth/decay rate changes."
[a] Number of excerpts is included for descriptive purposes only.		

Tracy used VMs and tasks because the tasks gave her students opportunities to explore content in a more dynamic way and with less restrictions than if the task had been done with paper and pencil. Students were able to offload the drawing aspects of creating the lawns to the VM. Additionally, students' engagement was increased because students' interests were piqued. The interactive environment of the VM helped to promote students' engagement with the task as well as reflecting on the mathematics through comparing solution strategies with their peers, thus promoting students' development of conceptual understanding.

On Daron's final survey, he said that he used the VM tasks because "students can discover and explore concepts in a dynamic way. [The VM tasks] allow for a deeper level of engagement and more efficient way to investigate deep mathematical ideas." Daron highlighted student explorationdue to the dynamic nature of the VM and student engagement as reasons why he used VMs and tasks.

Daron's comment about a "more efficient way to investigate" may be due to the ability to offload computational/drawing aspects of a task to the VM. Additionally, the statement could refer to the task providing some guidance to support students' exploration rather than students choosing how and what to explore. Due to the different possibilities of interpreting Daron's comment, this part of the comment was not coded.

Instructional guides. Besides the benefits to their students' developing understanding, teachers implemented VM tasks due to instructional support offered by accompanying instructional guides or questions. Twelve teachers mentioned purposefully looking for accompanying questions or instructional guides when searching for VMs to implement with their students. Many of the VM tasks that teachers chose to implement had accompanying guided questions or instructional guides to help students engage in the task and support their developing understanding. During Mark's final interview in May, he said,

> The thing that I got a LOT out of and it — I thought this was early on, I think we talked about it — was just having a study guide that kind of hits prior knowledge and then kind of guides the kids specifically through activities. And then kind of releases them to do, to solve problems based on what they, what they did with the virtual manipulative. The way the Gizmos work. I really like that progression. Umm, and so that's why I think it's easy to pick those, 'cause you know you're going to get that. Whereas other ones, you have to, you have to make stuff up. Umm, and it's not like that's super hard, once we're working together on things. [Capital letters used to denote emphasis in speech.]

Due to the belief that instructional guides were beneficial to students in activating prior knowledge and offering guidance, Mark was drawn to VMs that had accompanying questions or instructional guides. At times, the accompanying instructional guide or questions provided additional support related to how Mark may want to use the VM (e.g., launching a particular topic or providing more practice).

However, not all VMs had accompanying questions, especially VMs related to AP Calculus, AP Statistics, and at times Geometry. Many of the VMs related to these courses are categorized as the single-representation or multirepresentation environments. Therefore, to implement these VMs, teachers of these courses (i.e., Daron, Jake, and Mike) at times developed their own instructional guides to accompany a VM. During the January PD, Daron said,

> Some of the ones that I had been looking at don't really have a guide, so … I kind of keep these things in mind [i.e., the affordances in the task analysis framework] as I try to write a guide or put together some questions… to try to make sure as many of these were covered so it was a meaningful activity for them as well. Where otherwise you might just throw the questions out, you just, I think it allows you to think more like, "How do I move them up to those levels that we're trying to get them to?"

Therefore, Daron's efforts to develop instructional guides that further supported his students' learning and engagement were supported by a task analysis framework (Reiten, 2018), a tool in this activity system that was introduced during the PD.

How Teachers Implemented VM Tasks

While investigating how teachers implemented VM tasks during and after the course of the PD, two themes emerged. The first theme related to the instructional role for which the teachers used VM tasks. Therefore, how teachers implemented VM tasks (object) was influenced by their instructional goals and curricula (rules).

The other theme related to teachers' implementation efforts (object). Analysis of teachers' modifications of VM tasks (actions) drew from their discussions and responses on the Google Docs about the modifications they made. Both themes and teachers' modifications are discussed in the following subsections.

Instructional role. Teachers in the PD used VM tasks for a variety of purposes or instructional roles. The following subsection discusses the instructional role that the teachers reported VMs and tasks had within their classrooms over the course of the PD.

Table 3 highlights the different instructional roles teachers described related to their implementations of VM tasks. Teachers primarily reported using VMs and tasks to enhance their students' emerging understandings, as a reteaching tool, as a tool for enabling students to receive in-the-moment feedback, and as a tool for collecting information about student learning that could be used to inform future instruction. Unfortunately, as described in more detail in the next subsection, teachers did not often share their implementation efforts. Teachers implemented VMs and tasks more frequently than depicted in the initial data based on their final interviews and surveys.

Other instructional uses of VM tasks existed that are not included in the table. For example, teachers also reported using VMs to introduce or launch (Moyer-Packenham et al., 2008) and review particular concepts and units. However, these instructional uses only occurred three and four times in the dataset. Using VMs for demonstration purposes or modeling (Moyer-Packenham et al., 2008) was not a role that teachers identified. This lack of use is most likely due to the prevalence of technology in the classroom (e.g., most students had Chromebooks). Additionally, the PD focused on using VMs and tasks to promote student exploration as opposed to being used only for demonstration purposes. However, for teachers with more limited access to technology, using VMs for demonstrations or modeling with VMs may have been a means for enhancing student understanding or engagement.

During the January PD session, Daron explored a confidence interval for proportions VM from Flash & Math. He specifically was looking for a VM to use in his AP Statistics class that might enhance student understanding by supporting students' development of conceptual understanding through connecting the confidence interval and the graph of a distribution (see Figure 2).

During the spring months, Tracy often looked for VM tasks that allowed students to investigate and develop area formulas (e.g., calculating the area of a parallelogram) rather than simply being told the area formula. Therefore, Tracy, too, used VMs and tasks to enhance her students' emerging understandings.

Implementation efforts. The following subsection describes teachers' implementation efforts during the course of the PD. The findings are based on teachers' discussions and responses to the Google Docs during the PD sessions, as well as teachers' responses on the final survey and final interviews. Although the PD aimed to support teachers in teaching with VM tasks, there was no requirement for teachers to implement the VMs and tasks that they were critiquing and modifying during the PD session.

Table 3
Categories, Definitions, and Examples of Teacher Identified Instructional Roles of Virtual Manipulatives and Tasks

Definition	Number of Data Excerpts [a]	Example
Enhance: used to enhance students' understanding of concepts and skills through practice, communication, feedback, connecting representations and so forth	13	**Josh (FinalInt):** "I mean more to enhance some of the skills we're working on. We've used a lot of the Gizmos… at the end of that particular unit where we've talked about adding and subtracting fractions. And then it's, "*Well, let's do this Gizmo and see how it looks in a different light*. …So more to enhance and follow-up and build on some of the stuff that that they've already been learning about in class."
Feedback: provide feedback to students and/or teachers during the lesson	8	**Stan (OctPD):** "Well, we're getting feedback too. I think that's the MOST important."
Re-teach: used as a tool to re-teach a particular concept in a different way	11	**Tracy (FinalInt):** "I've used them with my Math Plus kiddos, more as another way to learn, another re-teaching. Because it's hard for them to, in their SECond math class of the day, to have to sit and do things pencil and paper. So, I try to use it more as some sort of instructional technology where they can see it. More, you know, just a different way…so it's different from what they did in their first math class of the day."
Solidify understanding: used as a tool to help solidify student understanding by having students interact with different representations.	7	**Josh (JanPD):** "Kids are used to it, they like it. …we've been using it more to solidify some of the things that we are doing in class [**Kelly:** Umm hmm]. When we did this activity, you know look on the Gizmo and try it on this."
Supplement curriculum: used as a tool to supplement curriculum by providing models of word problems, investigations and so forth	7	**Kelly (JanPDSummary):** Kelly said that her current textbook (i.e., Everyday Mathematics) does not have a lot of examples for what they are doing (i.e., adding fractions with unlike denominators). If they had the word problems in a ConceptuaMath module, then they could use the tools to model the problems, it would be helpful. Once they know how to use the tool, then it could be used to show different things.

[a] Number of excerpts is included for descriptive purposes only.

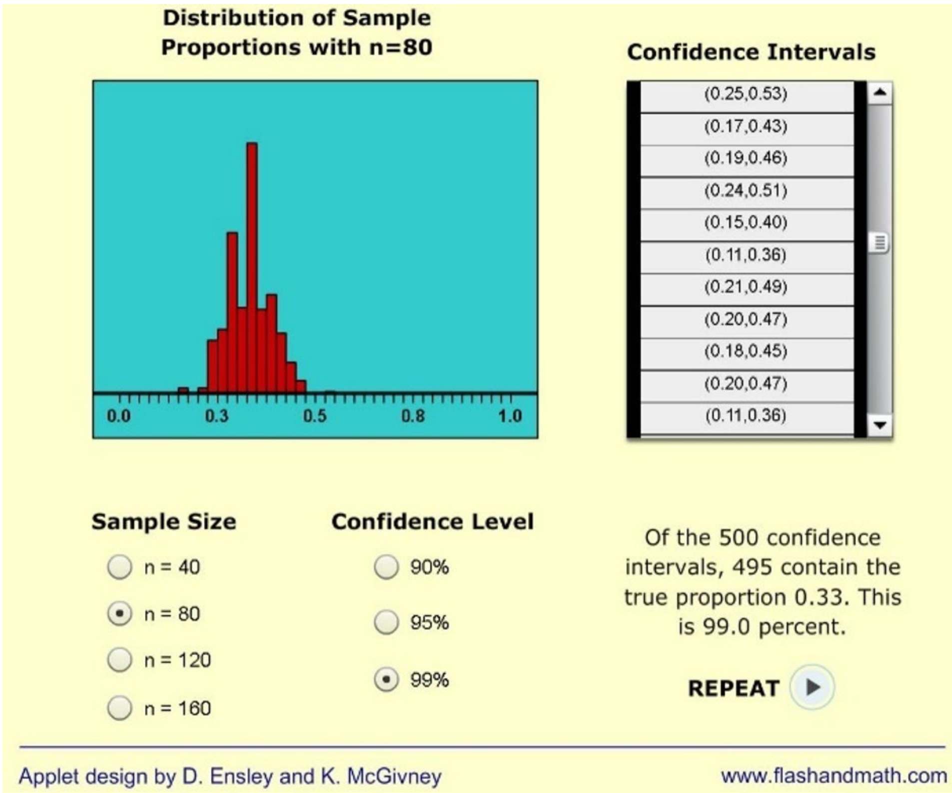

Figure 2. *Confidence Intervals for a Sample Proportion VM from Flash & Math (Note: This VM is no longer available through www.flashandmath.com.)*

At least 13 teachers implemented one or more VM tasks. Unfortunately, teachers did not always share their implementation efforts during the PD. For example, Tracy, Mark, and Josh mentioned during their final interviews that they had implemented at least 12 ExploreLearning gizmos alone, in addition to a few other VM tasks. Other common VM collections used by teachers in the PD included Shodor Interactivate, NCTM Illuminations, preconstructed GeoGebra worksheets, and Flash & Math, to name a few.

During the January PD session, Daron brought up his experience taking his AP Statistics students to the computer lab to do the Binomial Probability gizmo from ExploreLearning (https://www.explorelearning.com/index.cfm?method=cResource.dspDetail&ResourceID=126l; gizmo was not compatible with Chromebooks). Two days later, when he opened the gizmo during a class discussion, he and the class realized that it had now been converted to HTML5 (therefore, compatible with Chromebooks). According to Daron, after doing the gizmo, students had less questions about the binomial theorem compared to years past. Additionally, they were not relying on their calculators (rather, they were directly substituting into the formula) and were not making some of the common errors that previous students had made (leaving off the beginning part of the binomial formula). After the PD session, Daron emailed the following thoughts related to his implementation of the Binomial Probabilities gizmo:

> I think students were much more engaged in learning the binomial pattern. They were able to see where the pattern was coming up from the most basic situation to more complex examples. I think students were able to see the pattern in the factorials and Pascal's triangles quickly, then were able to adapt them to more difficult problems. I asked for student feedback and they said they really liked using the gizmo and it helped them to see the pattern clearly.

Daron's thoughts highlight how both he and his students found the VM task helpful for supporting students' developing understanding. This early positive experience implementing a VM task encouraged Daron to continue implementing VM tasks during the school year. By the end of the PD, Daron had implemented at least four VM tasks.

Modifications. Due to critiquing VMs and tasks, teachers began modifying VMs and tasks to better fit their learning goals and student needs. Additionally, critiquing and modifying VM tasks enabled teachers to focus on the value of the VM rather than a fun thing to try (Reiten, 2018).

Table 4 highlights the different types of modifications to VM tasks that teachers made during the course of the PD. Once again, the number of data excerpts related to each category are for descriptive purposes only and should be interpreted with caution. However, as displayed in the table, most of the modifications that teachers suggested related to helping students engage in the learning goal, creating an instructional guide, and addressing student needs.

Unlike earlier codes that were grounded in the literature, the codes related to the types of modifications arose from the responses of the teachers. Previous studies (e.g., Moyer-Packenham, et al., 2008; Moyer-Packenham, Salkind, Bolyard, & Suh, 2013) have investigated how teachers implemented VMs, but have not investigated how teachers modify VMs and tasks to implement in their classrooms. Due to the introduction of the guiding questions and task analysis framework supporting teachers' efforts to critique VMs and tasks, this study intentionally investigated how teachers modified VMs and tasks to implement with their students.

Often, teachers' modifications spanned more than one category. For example, as an intervention teacher, many of Tracy's students struggled with reading as well as mathematics (as also discussed in Vukovic, 2012). Therefore, Tracy's modification for the VM task that she was critiquing in December was to

> read each portion to the kids and have them answer before moving on. Give the kids screenshots of what each portion should look like. Possibly adjust the exploration guide to eliminate some of the language to make it a little more user friendly.

These modifications align with the instruction (reading each portion to the students), practical details (adding the screenshots), and student needs (eliminate some of the language to make it more user friendly) categories defined earlier.

Table 4
Categories, Definitions, and Examples of Teacher Modifications to VM Tasks

Definition	Number of Data Excerpts [a]	Example
Enhance: used to enhance students' understanding of concepts and skills through practice, communication, feedback, connecting representations and so forth	13	**Josh (FinalInt):** "I mean more to enhance some of the skills we're working on. We've used a lot of the Gizmos… at the end of that particular unit where we've talked about adding and subtracting fractions. And then it's, "*Well, let's do this Gizmo and see how it looks in a different light*. …So more to enhance and follow-up and build on some of the stuff that that they've already been learning about in class."
Feedback: provide feedback to students and/or teachers during the lesson	8	**Stan (OctPD):** "Well, we're getting feedback too. I think that's the MOST important."
Re-teach: used as a tool to re-teach a particular concept in a different way	11	**Tracy (FinalInt):** "I've used them with my Math Plus kiddos, more as another way to learn, another re-teaching. Because it's hard for them to, in their SECond math class of the day, to have to sit and do things pencil and paper. So, I try to use it more as some sort of instructional technology where they can see it. More, you know, just a different way…so it's different from what they did in their first math class of the day."
Solidify understanding: used as a tool to help solidify student understanding by having students interact with different representations.	7	**Josh (JanPD):** "Kids are used to it, they like it. …we've been using it more to solidify some of the things that we are doing in class [**Kelly:** Umm hmm]. When we did this activity, you know look on the Gizmo and try it on this."
Supplement curriculum: used as a tool to supplement curriculum by providing models of word problems, investigations and so forth	7	**Kelly (JanPDSummary):** Kelly said that her current textbook (i.e., Everyday Mathematics) does not have a lot of examples for what they are doing (i.e., adding fractions with unlike denominators). If they had the word problems in a ConceptuaMath module, then they could use the tools to model the problems, it would be helpful. Once they know how to use the tool, then it could be used to show different things.

[a] Number of excerpts is included for descriptive purposes only.

Tracy's modifications before implementing the VM task (her actions) demonstrate how teachers began to think carefully about how to implement VMs and tasks to promote their students' understanding (part of their community) rather than assuming the VM already met their student needs. While intentionally thinking about how her students might engage in the VM task that was initially selected, Tracy determined that her students may struggle with the reading level. Therefore, to promote student engagement in her learning goal (rule), rather than students' struggling with the reading, Tracy decided to modify the task so that she would read the instructional guide to the students and reduce some of the language demands on the students.

Discussion

Teachers have been implementing VMs and tasks for the past three decades. Despite the benefits to student engagement and understanding when teachers use VMs, studies indicate that teachers' use of VMs decreases as students progress through school (Moyer-Packenham & Westenskow, 2013). Unfortunately, few studies have investigated professional development opportunities aimed at supporting teacher's use of VMs (Driskell et al., 2016). Therefore, this study aimed to address both of these needs in the field by investigating why and how secondary mathematics teachers implemented VM tasks (object) while participating in a focused professional development opportunity.

Teachers in this study identified several benefits to students' emerging understanding as reasons why they implemented VM tasks. For example, teachers identified the visual nature of VMs (Moyer-Packenham et al., 2013a; Reimer & Moyer, 2005), opportunities for student exploration, and assisting in differentiating instruction (Bouck et al., 2015; Shin et al., 2017) as the most common benefits to instruction and student understanding.

Consistent with the field, teachers also cited students benefiting through the use of VMs and tasks due to the ability for students to offload computational aspects of a task (Moyer-Packenham et al., 2015), receive feedback (Edwards Johnson et al., 2012; Moyer-Packenham & Westenskow, 2013), practice problems (including extending beyond the VM), connect multiple representations (Moyer-Packenham et al., 2015; Moyer-Packenham & Westenskow, 2013), and encourage student engagement (Moyer-Packenham & Westenskow, 2013). Important to note, however, most of these benefits were identified for elementary students. Thus, this study extends the list of benefits in the current literature base to the secondary level.

New to this study, teachers also identified modeling (using or creating models) as benefit to students' emerging understanding and a reason why they chose to use VMs and tasks. Others have identified and discussed the remaining top eight teacher identified benefits. As attention to modeling increases, for example as teachers integrate the standards of mathematical practices (Common Core State Standards Initiative, 2010), this benefit of implementing VMs and tasks may become more prominent. It is not known, however, whether the identification of modeling is due to the teacher population (i.e., secondary teachers) or due to the increased focus of modeling in curricula.

These teacher-identified benefits also highlight an important factor when investigating why teachers choose to implement a VM task. That is to say, when investigating why secondary mathematics teachers implement a specific technology tool (in the case of this PD VM tasks), considering how they view their students' emerging understanding to be impacted is a mediating factor that needs to be considered. After all, teachers' students are part of their community (see Figure 1) and, hence, influence their efforts to teach with VM tasks.

If teachers do not think that their students will benefit from engaging with a VM task, then they usually will not implement the task. Daron's early positive feedback from students' interaction with the VM task from ExploreLearning encouraged him to keep looking for ways to integrate other VMs and tasks. Therefore, it is important to consider the influence teachers' early implementation efforts have on their future desire to implement new tools.

Due to the prolonged duration of the PD (October to May), teachers were able to see the impact the VMs and tasks had on student learning. Often, for changes in teachers' instructional practices to endure, teachers need to have opportunities to see improvements in student learning (Guskey, 2002).

During the PD, teachers identified instructional guides as being important components of VM tasks. In fact, teachers in this PD typically implemented VMs that had accompanying instructional guides or VMs that were in tutorial environments. Teachers in the PD stated that the instructional guides were beneficial because they provided guidance for their students and supported their students' developing understanding.

Since the focus of this PD was using VMs to promote student exploration as opposed to using VMs for demonstration only, it is not surprising that teachers gravitated to VMs in the tutorial environment or VMs that had accompanying instructional guides. However, this PD provided teachers with a tool (a task analysis framework) that supported their efforts to create instructional guides when guides did not already exist. Therefore the teachers in the PD were supported in developing VM tasks that aligned with how they thought their students' emerging understanding would best be supported.

Hence, when investigating why teachers choose to implement VM tasks, it is important to consider how teachers are supported in developing accompanying materials that they find helpful for supporting students' developing understanding. In this PD, the task analysis framework became an important tool that supported teachers' implementation of VM tasks (object).

The secondary mathematics teachers in this study identified implementing VMs for a variety of instructional roles. The findings of this study further support Moyer-Packenham et al.'s (2013b) findings that teachers in the PD were able to move beyond using VMs and tasks because they were a "fun thing to try."

Teachers commonly identified using VMs and tasks to enhance their instruction. However, teachers (see Table 3) also used VMs and tasks as a reteaching tool. For example, Tracy found it helpful to use VMs to help her students in the math intervention class to visualize the concepts in a different way than they had seen earlier in the day. The roles that teachers identified using VMs and tasks, however, may have been influenced by the emphasis in the PD, the access to technology (tool), and the teachers' curriculum (rule).

In this district, all grade 6-9 students had Chromebooks and some high school classes had class sets of Chromebooks. The following year, all students in grades 5-12 had Chromebooks. Thus, student access to technology was readily available. Most of the teacher-identified roles for using VMs and tasks are different from the uses identified by Moyer-Packenham et al. (2008; i.e., investigation, skill solidification, introduction, game, and other). It is not certain if the uses of VMs in this study were due to grades represented by the teachers, the access to technology, the focus or the PD, or the type of data collected (i.e., interviews, discussions, surveys, and responses to Google Docs vs. teachers' lesson summaries).

In terms of the activity system described earlier, these factors influencing how teachers used VMs and tasks relate to the rules (the curricula) and tools (Chromebooks) components of the system. Additionally, the district initiative (rules) to use technology as a means to support student engagement and understanding was a factor in how the teachers chose to implement VMs. For teachers with limited access to technology or without the expectation to implement technology for particular purposes, using VMs to launch or review units of study may be relevant to supporting their students' developing understanding. Furthermore, using VMs for demonstration or during whole class instruction may be other relevant uses even though these uses were not identified by teachers in this PD.

Division of labor is another component of the activity system that is important to consider when investigating why and how teachers implemented VM tasks. Teachers in the PD were supposed to sign up for the PD with their teaching partner to increase collaboration efforts. Only two teachers (Daron and Kelly) did not have someone in the PD with whom they could collaborate. The other teachers had one and sometimes more teachers with whom they could regularly collaborate. These teachers would jointly investigate potential VM tasks to implement with their students and work together to modify instructional guides to accompany the VM tasks.

Daron, however, worked alone to explore potential VM tasks. Additionally, many of the VMs for AP Statistics did not have accompanying instructional guides. Therefore, Daron spent time individually working to create instructional guides or guiding questions to accompany the selected VMs.

Another teacher, Tracy, often had one or two fellow teaching partners to work with during the PD. Tracy's opportunity to distribute work amongst her two collaborators may have contributed to the number of VM tasks that Tracy and her collaborators were able to implement during the PD (more than 12) compared to Daron (at least four).

Therefore, when investigating how and why teachers implement technology-based tasks, it is important to consider their collaborators and whether they can distribute the time to select and then modify the tasks to fit their learning goals and student needs. Teachers working individually or in isolation may need additional resources to make their selection of tasks more efficient. In this PD, the task analysis and guiding questions, repository of tasks and guidance related to where to search for potential tasks (Reiten, 2018) are example of such supports.

In general, teachers' experience and support resources to create instructional guides or guiding questions is something to consider. For example, many high school level VMs are single representation environments. For these environments, teachers (Daron, Jake, and Mike) typically created instructional guides or guiding questions to accompany the VM. Many of the VMs and tasks geared to the middle school level often had accompanying instructional guides or guiding questions. Teachers valued the role of the instructional guides and questions in supporting their students' emerging understanding and the time it could take to create the instructional guides, so the middle school teachers often implemented in their classroom only VMs in nonsingle-representation environments (see Mark's comment earlier about why it was easy to choose VMs from ExploreLearning).

Since the effectiveness of a VM or a VM task is contingent upon how it is implemented in the classroom (Ladel & Kortenkamp, 2016; Suh, 2016), it is important to consider how teachers are supported in creating instructional guides or guiding questions. Teachers need to be supported in modifying and developing VM tasks that support students' emerging understanding through communication, reflection, and connecting multiple representations. The linking of multiple representations as well as students' actions (i.e., simultaneous linking of VMs) can promote the development of conceptual understanding. Currently, however, minimal resources exist that may support teachers in creating instructional guides to accompany VMs in a single representation environment.

Although the PD could not affect the instructional time teachers had in their classrooms, the PD did provide teachers time to find, critique, and modify VM tasks specifically related to their upcoming units. As demands on teachers' time continue, integrating supported time within PD for teachers to find and develop VM tasks directly connected to their curriculum can help to support teachers' integration efforts by strengthening the connection between the PD and teachers' instructional practices (Wilson, 2008).

Anecdotally, structuring teachers' work time during the PD proved to be more productive than teachers choosing how to use the work time that was given. Allocating time to search and select tasks, then modify and prepare instructional guides for use with their students, then providing discussion/reflection time was more productive compared to when teachers were not given structure for how to use the time given. Additionally, it is important to note that teachers came to each PD session with a predetermined learning goal. This learning goal helped to focus their searching efforts and therefore use the allocated time more efficiently.

Implications for Teacher Education

Although VM tasks were new to over half of the PD participants, for a majority of the teachers in the PD, VM tasks became a regular tool that they implemented in their classrooms. Therefore, the findings indicate that secondary mathematics teachers can be supported to implement VM tasks; hence, potentially addressing the decrease in VM use as students progress though the grades.

Improvement in teachers' implementation of VM tasks, is a process (not an event) and one that must be continually supported as teachers strive to improve student understanding (Guskey, 2002). Future work needs to be done to investigate how teachers, especially secondary mathematics teachers, implement VMs and tasks to support students' development of conceptual understanding; for example, investigating teachers implementing the tasks, as well as investigating the instructional guides and guiding questions that teachers use to accompany VMs. When investigating how and why teachers use a particular technology tool, it is important to consider how they are supported in doing so; for example, considering how they receive training in the tool (PD) and support resources (e.g., repository or VMs or task analysis framework).

Additionally, it is important to acknowledge the influence that various components of teachers' activity systems have on their implementation efforts when investigating why and how teachers implement technology tools and tasks. Teachers most frequently modified VMs and tasks to better support their learning goals and student needs (modifying/developing instructional guides, modifying the implementation strategy, etc.). Students are an important component of teachers' activity systems and influence why and how teachers' implement particular technology tools and tasks.

Since students' achievement levels are related to teachers' experience using the manipulatives (Moyer-Packenham et al., 2013b), how teachers are supported in engaging with VMs is important. Therefore, it is not enough to only provide access to a tool or make teachers aware of a particular tool. Rather, teachers need to be supported in learning how to critique technology based tools and tasks so that they are able to modify and/or design quality opportunities for their students to engage with the tools and tasks. Quality opportunities are ones in which students' emerging understanding is supported through communicating, reflecting, and connecting multiple representations. Therefore, PD opportunities are needed that provide teachers opportunities to interact with technology tools situated within quality opportunities.

VMs and tasks (as well as other technology tools and tasks) can be used to address issues of inequity and support students in developing rich understandings of mathematical concepts. However, to realize the full potential of VMs (or any technology tool for that matter) teachers need to be supported in critiquing technology tools and tasks based on their learning goals and student needs. Additionally, teachers need to be supported in moving beyond choosing a particular tool or task because it is a *fun thing to try*. Focused PD opportunities, similar to the one investigated in this study, are one means to support and prepare teachers to use technology in an effective and innovative manner.

Linking technology tools and tasks directly to teachers' curricula units helps to strengthen the connection to teachers' practice and the PD. How teachers are supported to engage in and implement (new) technology tools has the potential to transform the learning opportunities for their students. As the expectations for teachers to implement technology tools and tasks continues to increase, how we support teachers is a critical issue.

Notes

[a] Through their review of 580 lesson summaries of 95 K-8 teachers, Moyer-Packenham, Salkind, & Bolyard (2008) identified "seven categories describing how mathematical tools were used in lessons" (p. 207). The categories were (a) investigate concepts, (b) skill solidification, (c) introduce, (d) game, (e) aid, (f) model, and (g) extend. The investigate concepts (which relates to exploration) and extend (which relates to differentiation) codes in why teachers implemented VM tasks.

[b] Fido's Flower Bed (https://www.explorelearning.com/index.cfm?method=cResource.dspDetail&ResourceID=1011) is a gizmo from ExploreLearning exploring perimeter and area. Using squares representing sod, students are supposed to construct lawns and then fence the lawn to explore perimeter and area. They can use prebuilt lawns of 36 squares to explore lawns containing the same area but different perimeters or create their own lawns by clicking where they want to place the sod.

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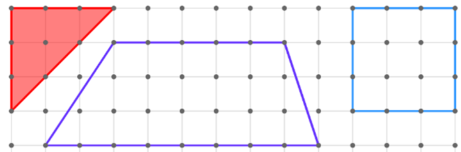
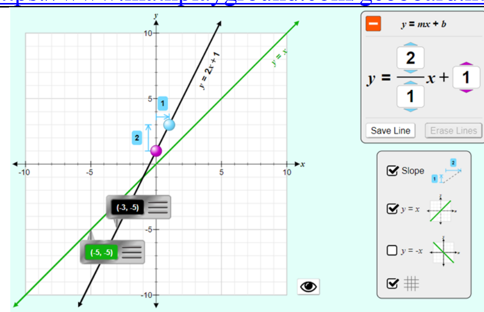
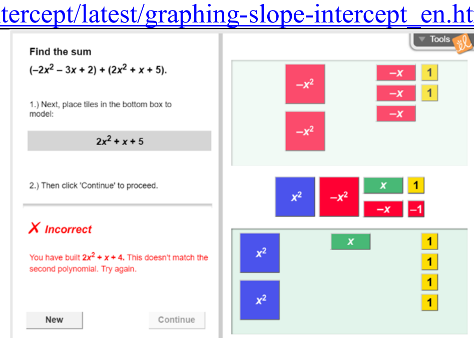
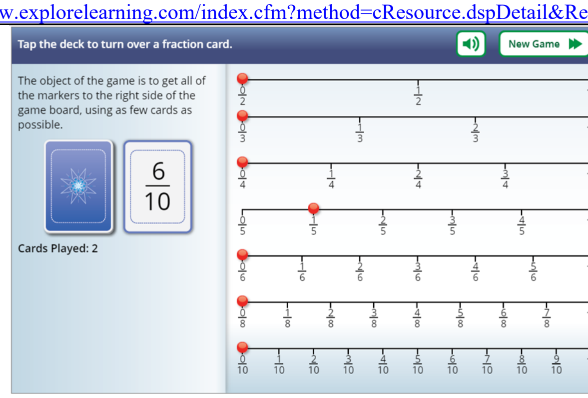
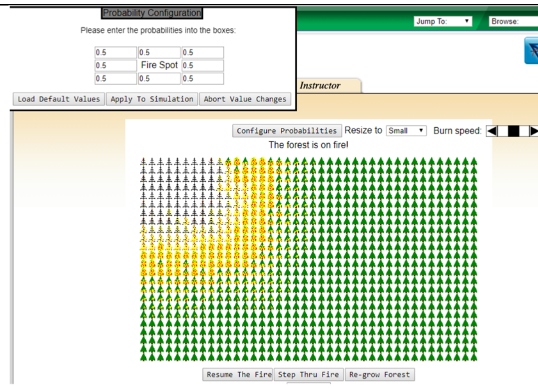
Appendix A
Examples of VMs Used by Teachers During the Professional Development Opportunity

VM Environment	Creator	Example
Single Representation	MathPlayground	 Geoboard (https://www.mathplayground.com/geoboard.html)
Multi-Representation	PhET Interactive Simulations	 Graphing Slope-Intercept (https://phet.colorado.edu/sims/html/graphing-slope-intercept/latest/graphing-slope-intercept_en.html)
Tutorial	ExploreLearning	 Addition of Polynomials (https://www.explorelearning.com/index.cfm?method=cResource.dspDetail&ResourceID=97)
Gaming	NCTM Illuminations	 Fraction Game https://www.nctm.org/Classroom-Resources/Illuminations/Interactives/Fraction-Game/

Simulation	Shodor Inter*activate*	 Directable Fire (http://www.shodor.org/interactivate/activities/DirectableFire/)

Appendix B
Task Analysis Framework

Affordances	Descriptions
N/A	Task is primarily a technology task with no focus on mathematics.
N/A	Virtual manipulative does not have mathematical fidelity required to respond to the prompts.
A	Task prompts students to recall a mathematical fact, rule, formula, or definition.
B	Task prompts students to report information from the virtual manipulative or consider mathematical concepts, processes, or relationships in the current display. The student is not expected to provide an explanation.
C	Task provides opportunities for students to explain the mathematical concepts, processes, or relationships in the current display.
D	Task provides opportunities for students to make predictions and then test their predictions using the virtual manipulative.
E	Task provides opportunities for students to connect multiple representations of a mathematical concept (e.g., graphical, algebraic, and tabular representations of a relation).
F	Task provides opportunities to check students' understanding of mathematical concepts, processes, or relationships. Task may provide minimal feedback to the student based on specific errors.
G	Task provides opportunities for students to go beyond the current display by considering multiple examples to generalize mathematical concepts, processes, or relationships.
H	Task supports students' exploration through manipulation of the display that may surprise one exploring the relationships represented or cause one to refine thinking based on themes within the surprise (e.g., addressing a common student misconception).

Note. This framework is intended to help teachers better critique and develop tasks aimed at promoting students' development of conceptual understanding of mathematics through reflection and communication (Hiebert et al., 1997), as well as through using and connecting mathematical representations (NCTM, 2014). Descriptions below are not necessarily in a hierarchical ordering nor are they mutually exclusive. Portions of the table below are adapted from Trocki (2014) and Sinclair (2003).

Guiding Questions

1. What is your learning goal (for your students)?
2. How might your students struggle during this exploration?
3. How might your students benefit from engaging in this exploration?
4. Thinking about your learning goal, what is one modification you would make so that the exploration better fit the needs of your students?
    - *Why might this modification help your students engage in the learning goal?*

Appendix C

Please respond to the prompts below. The prompts are intended to help you modify the VM tasks with your students in mind.

Choose one of the tasks listed in the **VM Tasks** folder. Or, if you have a topic in mind that is not listed, you may use the links on the **VM Collections** page to find a new task.

Which activity did you choose to modify? ____________________________________

Complete the student handout for your task (if one exists). Available instructional guides are located in the various folders. If you download a new one, please share it with me. If no handout exists, spend some time exploring the VM and then respond to the questions below.

1. What is an essential question you would use this activity to explore?

2. Where do you think your students might struggle in this exploration?

3. How might your students benefit from engaging in this exploration?

4. Thinking about the essential question, what is one modification you would make so that the exploration better fit the needs of your students? Why might this modification help your students engage in the essential question?

Now, use the framework that we have been talking about (i.e., the **Task Analysis Framework_v3**) to help you think about how to modify the task. The updated framework is posted in the Google Folder.

1. Which prompts of the framework apply to the task you chose? **What from the task supports your claim(s)**?

Now, use the framework to help you modify the task to better fit the needs of your students. Please email or share with me the document that you create. You may want to shorten the activity or modify some of the prompts to better align with your learning goals and the needs of your students. Or, maybe you will need to create an instructional guide.

After modifying the task, which prompts of the framework apply to the task you chose? **What from the task supports your claim(s)**?